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Review

Advances in peridynamics modeling of deformation and fracturing of brittle geomaterials

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ABSTRACT

Peridynamics (PD) is an emerging method that establishes a theoretical framework based on non-local theory to describe material mechanical behavior with spatial integral equations. It gives a unified expression of the medium including state transformation and characterization in different scales. It is showing great potential for evaluating the complicated mechanical behaviors of brittle solids. In the past two decades, peridynamics has been showing its great potential and advantages in modeling crackings of brittle materials although there are many challenges. The present paper summarizes firstly the theoretical framework and advantages of peridynamics for modeling fracturing. It introduces then the theoretical improvements to address challenges of peridynamics in modeling brittle solid crackings including the release of Poisson ratio limit, different fracture criteria, contact-friction models, coupled constitutive models, and computing accuracy. Afterward, the extension of peridynamics is introduced to the coupled modeling with the other methods such as finite element method, phase field method, and particle-like method before its applications in static and dynamic cracking as well as those under impacts. Meanwhile, some contents that require further exploration are briefly summarized. Finally, the blind spots and future development of peridynamics are analyzed and discussed for the deformation and fracturing modeling of brittle geomaterials.

1. Introduction

In general, for engineering materials with micro-cracks, such as rocks, concrete, ceramics, and other brittle materials, crack initiation is a manifestation of initial damage generation[1,2]. Under external loading, a local zone of deformation is first formed in an internal region of the material. Then cracks develop and expand within the material. For example, the tip stress concentration of an internal crack in rock-like materials is the main cause of cracking[3]. In this process, the material on both sides of the crack gradually changes from a continuous state to a discontinuous state, thus gradually exhibiting the characteristics of a discontinuous medium. The process is complicated and multifaceted. It includes the transition from continuous to discontinuous media, from elastic-plastic to damage fracture mechanics, and from microscopic to macroscopic models[4,5]. In the research process, the stochastic primary discontinuous structure and the micro-size effect in the initial state transition greatly increase the complexity of the problem. The numerical simulations are becoming more and more important in analyzing the crack initiation-expansion process [6].

Compared with lab experiments and in-situ observations, numerical methods have advantages such as low costs and short durations. With the development of computer technology, numerical simulation has also been greatly promoted in its capabilities[7]. At present, numerical simulation methods have become common and indispensable research tools for studying complicated mechanical responses in various practical engineering problems[8,9].

In computational fracture mechanics, numerical methods can be broadly classified into three categories respectively of continuous methods, discontinuous methods, and continuous-discontinuous methods (coupled or combined methods)[3]. In general, classic

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continuous methods are based on continuous mechanics such as the Finite Element Method, Finite Difference Method and Boundary Element Method. They are competent in simulating the distribution of the stress, strain, and plastic zone but incapable of simulating the crack initiation-expansion process. Meanwhile, they are often plagued by mesh dependence problems caused by prefabricated cracks. Classic discontinuous methods (e.g., Discrete Element Method; Manifold Method; Discontinuous Deformation Analysis, etc.) are based on mathematical ideas, such as block theory and topology theory. They can simulate well the motion and interaction processes of discontinuous media. However, parameters such as contact stiffness are often introduced. It may cause accuracy loss in calculating the stress and strains in the material elastic stage.

Then, the continuous-discontinuous methods have been developed rapidly[10] to combine the advantages of the above two methods, such as the combined Boundary Element Method/Finite Element Method [11], Discrete Element Method/Finite Element Method[12], and Numerical Manifold Method[13]. However, it is often necessary for these methods to introduce some strength criteria and contact models for the formation and evolution of discontinuous media. The theories of combining the methods are incomplete and under development and improvement. The challenges are largely due to the mismatch or contradiction between the theoretical foundations (e.g., continuous media mechanics, etc.) and quasi-simulated physical phenomena (e.g., discontinuity problems, etc.). As a result, they will exhibit inadaptability in terms of calculation accuracy and efficiency. On this point, peridynamics (PD), as a new method, establishes numerical models based on the non-local theory and describes mechanical behavior by spatial integral equations was developed[8].

As an emerging theory for dealing with discontinuous problems, PD method can naturally simulate the medium transformation process as well as the evolution process of discontinuous medium[14,15]. At present, the advantages of PD have greatly promoted its development in studying material fracture behaviors, and a series of research results have been achieved as shown in Fig. 1. At present, the research of PD can be divided into two aspects. The first is the continuous development and improvement of the PD theoretical system in theoretical models and analytical solutions, which verifies its effectiveness and improves its universality. The second is the continuous improvement and breakthrough of applying PD to simulating material damage and fracture. It mainly includes the study of static/quasi-static crack propagation, dynamic crack propagation, impact fracture, macro-scale large deformation, and micro-scale phase transformation dynamics of brittle materials. In addition, combining the PD with some other classical methods is also in progress to meet the needs of practical engineering simulation.

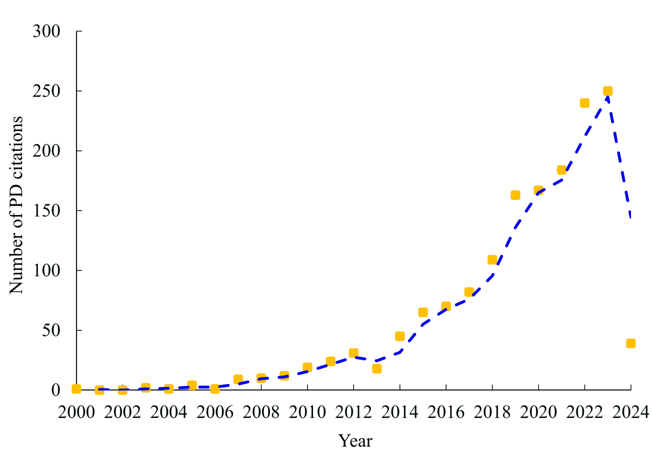

Fig. 1. Number of PD citations per year (the data obtained from Web of Science): the solid yellow square is the number of citations, and the blue dotted line is the fitting curve.

The PD is still in its infancy and showing its potential in new cutting edge problems although it has been developed for more than two decades. This paper presents a systematic review and summary of the PD development in brittle material damage. First, the basic principles, classification, and advantages of PD are briefly introduced in Section 2. Then, a systematic review and summary of specific aspects of the two above aspects is presented in Sections 3 and 4. It includes research progress in various research aspects. For example, Poisson's ratio limitation, fracture criteria, contact-friction model, constitutive model, numerical solution accuracy, applicability problems, and coupling methods. Finally, preliminary analysis and discussion about the blind spots and development directions of PD methods are presented in Section 5.

2. Basic theory and advantages of Peridynamics

In 2000, Dr. Silling of Sandia National Laboratories, USA, completely reconstructed the traditional continuum mechanics theory with the displacement space integral equation, and finally proposed the basic idea of PD[16]. As shown in Fig. 2, if an object occupies a certain spatial domain Ω, at a certain time t, there is an interaction between any material point $\mathbf{x}$ in Ω and other material points $\mathbf{x}'$ in a certain range of space $H_{\mathbf{x}}$. This range is referred to as the horizon, usually, a circular or spherical domain, and the radius of the horizon is δ. All other material points $\mathbf{x}'$ in the horizon are referred to as family material points of this material point $\mathbf{x}$ (this $\mathbf{x}$ is referred to as the central material point), and satisfy the following relation[16]:

$$\mathbf{x}' \in H_{\mathbf{x}} : d_{\mathbf{xx}'} \leqslant \delta \tag{1}$$

where $d_{\mathbf{xx}'}$ is the distance between material points.

Although the force between the material points is referred to as the force density, unlike classical continuum mechanics, its dimension is FL^{-6}. The force density depends primarily on the initial relative position vector, the relative displacement vector, and the material parameters. The interaction between the material points is reflected by the force density function $\mathbf{f}$, i.e., the constitutive force function. It describes the relationship between internal forces and deformation. It is the basis of the motion equation of PD, which satisfies Newton's second law form:

$$\rho \ddot{\mathbf{u}}\left(\mathbf{x}, t\right) = \int_{H_x} \mathbf{f}(\mathbf{u}(\mathbf{x}, t), \mathbf{u}'(\mathbf{x}', t), \mathbf{x}, \mathbf{x}', t) dV_{\mathbf{x}'} + \mathbf{b}\left(\mathbf{x}, t\right) \tag{2}$$

where ρ is the material density; $\mathbf{u}$ is the displacement of the material point x, $\ddot{\mathbf{u}} = (\partial^2 \mathbf{u}(\mathbf{x},t))/\partial t^2$; $V_{\mathbf{x}'}$ is the volume of the family material point $\mathbf{x}'$; $\mathbf{b}$ is the external load on the object per unit volume.

In general, the interaction between two material points is described

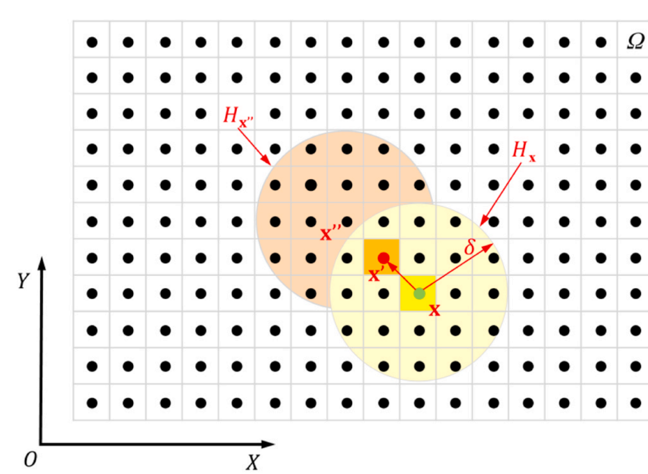

Fig. 2. Two-dimensional discretization system and horizon of peridynamics[17].

by the concept of a bond. The definition of the relative position vector $\xi = \mathbf{x}' - \mathbf{x}$ and the relative displacement vector $\eta = \mathbf{u}'(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)$ can be well used to characterize the degree of bond length variation $s(\eta, \xi, t)$, and they allow the PD motion equations to be simplified, as follows:

$$s\left(\boldsymbol{\eta}, \boldsymbol{\xi}, t\right) = \frac{\|\boldsymbol{\eta} + \boldsymbol{\xi}\| - \|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|} \tag{3}$$

$$\mathbf{f}(\mathbf{u}(\mathbf{x}, t), \mathbf{u}'(\mathbf{x}', t), \mathbf{x}, \mathbf{x}', t) = \mathbf{f}\left(\eta, \xi\right)\frac{\boldsymbol{\eta} + \boldsymbol{\xi}}{\|\boldsymbol{\eta} + \boldsymbol{\xi}\|} \tag{4}$$

where $\|\boldsymbol{\xi}\|$ and $\|\boldsymbol{\eta} + \boldsymbol{\xi}\|$ are the lengths of the bond at the initial and deformation moments, respectively. The critical degree of the bond length variation is a good criterion to reflect the occurrence of material damage and fracture, and PD incorporates it into a unified representation of the force density function so that it finally gains the unique advantage of simulating the natural initiation-propagation process of cracks:

$$\mathbf{f}\left(\boldsymbol{\eta}, \boldsymbol{\xi}\right)\frac{\boldsymbol{\eta} + \boldsymbol{\xi}}{\|\boldsymbol{\eta} + \boldsymbol{\xi}\|} = cs\left(\boldsymbol{\eta}, \boldsymbol{\xi}, t\right)\mu\left(\boldsymbol{\eta}, \boldsymbol{\xi}, t\right) \tag{5}$$

$$\mu\left(\boldsymbol{\eta}, \boldsymbol{\xi}, t\right) = \begin{cases} 1 & s(\boldsymbol{\eta}, \boldsymbol{\xi}, t') < s_0, 0 < t' < t \\ 0 & else \end{cases} \tag{6}$$

where $\mu(\boldsymbol{\eta}, \boldsymbol{\xi}, t)$ is an eigenfunction describing the bond state, $\mu(\boldsymbol{\eta}, \boldsymbol{\xi}, t) = 1$ which means the bond is in the state of connection and otherwise it is broken; s_0 is the critical degree of bond length variation; c is the PD micro-modules and can be translated to the shear modulus G in classical continuum mechanics, which indicates the potential connection of PD to classical continuum mechanics.

The bond-based peridynamics (BB-PD) is the original peridynamics. It has received a lot of attention due to its simple and clear physical meaning and high computational efficiency[18,19]. After its continuous development and improvement, a complete computing system has been formed. However, its limitations gradually appear and are manifested in two aspects. Firstly, since the **f** is completely based on the interaction of a single material point pair, the scope of describing the material constitutive characteristics is limited (Poisson ratio limitation, see later) [20]. In the second aspect, the combination of BB-PD and traditional theory is not close enough. The traditional theoretical physical parameters (e.g., stress and strain, etc.) are not involved. It has a great influence on numerical modeling and variable analysis.

To overcome the above limitations and extend the application scope, state-based peridynamics (SB-PD) has been developed rapidly[21]. When the **f** considers the deformation of all bonds connecting two material points, it is SB-PD. The term 'state' is a mathematical concept proposed by Dr. Silling's team. It can be understood as a mapping from a vector space to a tensor ensemble. The state is the same operation on all the bonds (i.e., the set of vectors) in the horizon. It is convenient for expressing the set of all bonds in the horizon and the operations about this set. E.g., let $\underline{M}$ denote a vector state such that $\underline{M}$ <**x**-**x**'> is a unit vector pointing from the deformed position of **x** toward the deformed position of $\mathbf{x}_0$. $\underline{M}$ <**x**-**x**'> and deformed form **y** can be used to calculate the interaction of two material points in the deformed configuration.

The SB-PD can be further divided into ordinary and non-ordinary ones, depending on whether the direction of the force state coincides with the direction of the morphology. As shown in Fig. 3, the two directions are consistent in ordinary SB-PD, and it can be understood as a special case of non-ordinary SB-PD. For a linear elastic isotropic material, the force density vector $\underline{T}$ is the correlation function of the strain energy density $w(\mathbf{x})$, and the two must meet the following requirements [22]

$$\underline{T}\left(\mathbf{u}'(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t),\ \mathbf{x}' - \mathbf{x},\ t\right) \sim \frac{\partial w(\mathbf{x})}{\partial(|\boldsymbol{y}' - \mathbf{y}|)}\frac{\boldsymbol{y}' - \mathbf{y}}{|\boldsymbol{y}' - \mathbf{y}|} \tag{7}$$

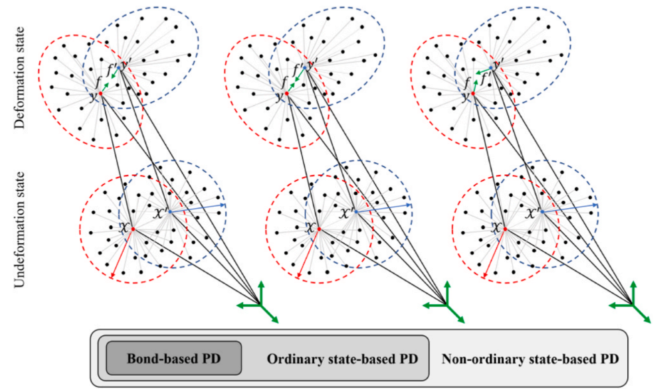

Fig. 3. Peridynamics classification[22].

and

$$\underline{T}\left(\mathbf{u}(\mathbf{x}, t) - \mathbf{u}'(\mathbf{x}', t), \mathbf{x} - \mathbf{x}',\ t\right) \sim \frac{\partial w(\mathbf{x})}{\partial(|\mathbf{y} - \boldsymbol{y}'|)}\frac{\boldsymbol{y}' - \mathbf{y}}{|\boldsymbol{y}' - \mathbf{y}|} \tag{8}$$

where, due to the interaction between material points, a scalar-valued micro potential is

$$w_{(k)(j)} = w_{(k)(j)}\left[y_{(1^k)} - y_{(k)}, y_{(2^k)} - y_{(k)}, \cdots\right] \tag{10}$$

and

$$w_{(j)(k)} = w_{(j)(k)}\left[y_{(1^j)} - y_{(j)}, y_{(2^j)} - y_{(j)}, \cdots\right] \tag{11}$$

where j and k are material point numbers.

Then, the force state in SB-PD is given by

$$\underline{T}\left(\mathbf{x},\ t\right) < \mathbf{x}' - \mathbf{x} >= \left(\frac{2ad\delta}{|\mathbf{x}' - \mathbf{x}|}\theta\left(\mathbf{x}, t\right) + bs\left(\boldsymbol{\eta}, \boldsymbol{\xi}, t\right)\right)\frac{\boldsymbol{y}' - \mathbf{y}}{|\boldsymbol{y}' - \mathbf{y}|} \tag{12}$$

where a, b, and d are PD parameters and $\theta(\mathbf{x}, \mathrm{t})$ is the PD dilatation term.

The direction assumption can be disregarded in the non-ordinary SB-PD by allowing force density in arbitrary directions. So, it is essential to explicitly ensure the conservation of angular momentum, for the non-ordinary SB-PD, the following relationship must hold[22]

$$\int_{H_x}\{(\boldsymbol{y}' - \mathbf{y}) \times \underline{T}(\mathbf{x}, t) < \mathbf{x}' - \mathbf{x} >\}dV' = 0 \tag{13}$$

Furthermore, taking the correspondence model, the force density vector can be related to stress as

$$\underline{T} < \mathbf{x}' - \mathbf{x} >= \underline{w}(|\mathbf{x}' - \mathbf{x}|)\overline{\boldsymbol{P}}\cdot\boldsymbol{K}(\mathbf{x})\cdot\xi \tag{14}$$

where $\underline{w}(|\mathbf{x}' - \mathbf{x}|)$ is influence function; $\overline{\boldsymbol{P}}$ is the first Piola-Kirchhof stress, and it is related to the second Piola-Kirchhof stress $\boldsymbol{S}$

$$S = 2\frac{\partial W}{\partial \boldsymbol{E}} = \boldsymbol{\Psi} : \boldsymbol{E} \tag{15}$$

where $\boldsymbol{W}$ is elastic strain energy, $W = 1/2(\boldsymbol{E} : \boldsymbol{\Psi} : \boldsymbol{E})$, $\boldsymbol{\Psi}$ is the fourth-order elastic tensor coefficient, and $\boldsymbol{E}$ is the Green-Lagrange tensor. Meanwhile, $\boldsymbol{K}(\mathbf{x})$ is the shape tensor defined as

$$\boldsymbol{K}(\mathbf{x}) = \left[\int_{H_x}\underline{w}(|\xi|)(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\mathrm{d}V_\xi\right]^{-1} \tag{16}$$

Incorporating $\boldsymbol{K}(\mathbf{x})$, a non-local deformation gradient can be defined as

$$\boldsymbol{F}(\mathbf{x}) = \left[\int_{H_x}\underline{w}(|\boldsymbol{\xi}|)(\underline{\boldsymbol{Y}}(\boldsymbol{\xi}) \otimes \boldsymbol{\xi})\mathrm{d}V_\xi\right]\cdot\boldsymbol{K}(\mathbf{x}) \tag{17}$$

Therefore, the concepts such as stress and strain are considered in non-ordinary SB-PD, which makes it closely related to classical continuum mechanics.

It should be pointed out that the SB-PD is only a solution to overcome the limitation of BB-PD. There are still some aspects to be improved with the vigorous development of SB-PD. For example, the approximate deformation gradient is less dependent on the deformation of the bond, so there is a zero-energy mode in the non-ordinary SB-PD[23], i.e., the hourglass problem, in which there are few solutions in PD. It can cause numerical oscillations and result in reduced accuracy. Meanwhile, due to the advantages of BB-PD, the improved BB-PD methods and models are being proposed.

Both BB-PD and SB-PD have two main advantages over classical continuum mechanics, which are as follows:

(1) Description of material mechanical behavior by spatial integral equation

In classical continuum mechanics, most of the motion equations take different forms of displacement derivation. However, when the displacement is discontinuous, the displacement derivation will lose its meaning. The spatial integral equation of PD is no longer presented in the form of a traditional stress-strain relationship. The calculation form of discontinuity and continuity is summarized into a set of equations. This equation is still defined at discontinuities. It no longer assumes continuity of the displacement field, and no longer involves derivation. This means that the displacement field no longer affects the solution of the entire equation. Discontinuous situations (e.g., the crack initiation-expansion process, etc.) can naturally appear directly with the motion trajectory of material points. Therefore, PD can avoid the ill-conditioned characteristics of the continuum mechanics model and realize a unified description of the mechanical behavior of continuum and discontinuous media.

(2) Theoretical framework based on non-local theory

In classical continuum mechanics, locality is a fundamental assumption[24]. In a local theoretical model, a material point only exchanges mass, momentum, and energy with material points in its vicinity. It will cause the stress state of a material point to depend only on the deformation state of the adjacency material point. Meanwhile, there is no internal length parameter in the theoretical model to distinguish the length size. It makes the hypothesis's applicability in the micro and macro scopes questionable. However, the PD as a meshless method has a new non-local model that considers the remote interaction, as shown in Fig. 4. PD establishes the relationship with different length scales through different values of the horizon. Results have shown that non-local models can deal with a wide range of wavelength valleys[25]. This non-local model can not only reflect the macroscopic effect more realistically, but also better simulate the microscopic effect of molecular and atomic size.

It should be noted that, as shown in Fig. 5, both PD and classical continuum mechanics are theoretical formulations defined on a continuum medium R, except that the former is divided into material points with finite volume. PD and Molecular Dynamics (MD) are both meshless methods, and PD refers to some concepts from MD to some extent (e.g., the radius of the horizon and the interaction between material points, etc.). However, MD is mainly devoted to the analysis of interactions between discrete entities, i.e., at the atomic and molecular scales. If analyzed from the perspective of PD numerical realization, it can be regarded as a continuous version of MD[26]. In theory, MD is the most detailed and realistic method for predicting material damage and fracture behavior[27], but it is highly susceptible to computational resources and simulation time. Although MD simulation capability is improving continuously with the development of computer hardware/software technology (e.g., Kadau et al.[28] simulated a square copper sheet (a side length of 1.56 microns) by 3.2×10^{8} atoms), the scale level is still very small for actual engineering structure. In addition, although PD is a meshless method, it is different from classical meshless methods, e.g., Smoothed Particle Hydrodynamics, SPH; Galerkin Method, GM, etc. This is mainly related to the integral motion equation of PD.

In summary, in a non-local theoretical framework, PD based on the spatially integral equation combines the advantages of MD and meshless methods. It can not only avoid the singularities of traditional numerical methods based on the continuity assumption and the differential equation but also break through the limitations of MD in calculation and time scale. This method can not only naturally simulate the transformation process from continuous to discontinuous media, but also simulate the further evolution of discontinuous media. It has become a potential numerical simulation method to evaluate the mechanical response of brittle materials.

3. Improvement of peridynamics theoretical framework

3.1. Poisson ratio limitation

As the original peridynamics, the BB-PD is relatively mature and has high universality (e.g., rock materials, glass materials, ceramic materials, etc.). However, for the BB-PD, it is impossible to distinguish volumetric strain and deviatoric strain by only considering the bond deformation connecting two material points. Different from classical continuum mechanics, which uses two independent elastic parameters (elastic modulus and Poisson ratio) to characterize isotropic linear elastomers, BB-PD reduces two independent parameters to an

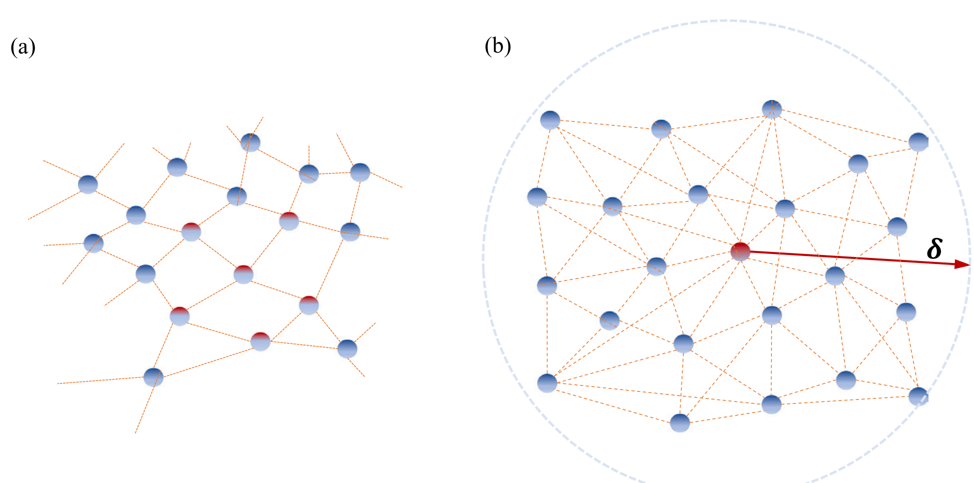

Fig. 4. Peridynamics and traditional continuum mechanics models[25]: (a) is a local interaction model, and (b) is a long-range non-local interaction model.

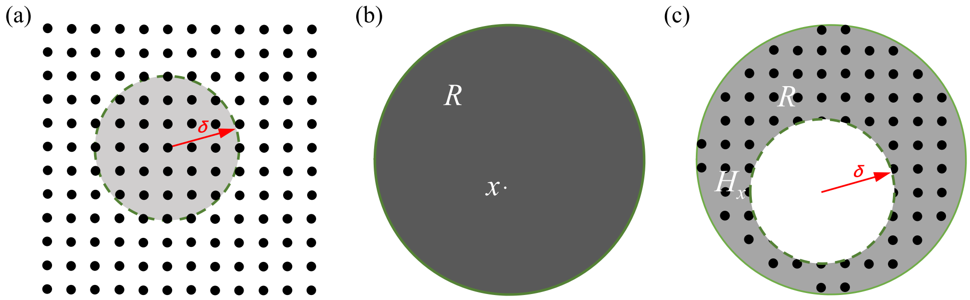

Fig. 5. Schematic models of molecular dynamics, classical continuum mechanics, and peridynamics[22]: (a) is Molecular Dynamics, (b) is Classical continuum mechanics, and (c) is Peridynamics.

independent micro-modulus parameter (micro module). This will inevitably lead to a certain elastic parameter being solidified into the constitutive model. It is the problem that BB-PD has been controversial and urgently solved in its development process, the Poisson ratio limitation[29].

In BB-PD, the Poisson ratio is solidified as a constant value(it is 1/3 in plane stress conditions, 1/4 in plane strain and three-dimensional stress conditions). This leads to the fact that BB-PD can only be applied to materials with the above-mentioned Poisson ratio. In other words, the Poisson ratio limitation greatly limits the further development of BB-PD. For this reason, to retain the physical meaning simplicity and the numerical calculations stability of traditional BB-PD, some scholars have alleviated the Poisson ratio limitation by classical theoretical reconstruction scheme[20,30–36], bond rotation scheme[37–42], new bond type scheme[19,43,44] and multi-parameter description scheme [45–50]. The common advantage of these schemes is that only added the tangential interaction between two material points, so the original advantage can be retained to the greatest extent.

(1) Classical theoretical reconstruction scheme

The classical theory reconstruction scheme refers to introducing certain classical theories into the BB-PD theory, and constructing certain physical quantities belonging to BB-PD from various perspectives, to break through limitation.

Braun M. et al.[30] proposed a two-dimensional discrete model and a special discretization by Navier's equation, which eventually extended the Poisson ratio range. The numerical example shows that the Poisson ratio could be set to 0 ~ 0.49. EKIZ E et al.[31] proposed a pure geometric description method for the Poisson ratio. The Poisson ratio is calculated by minimizing the internal energy density. Finally, the Poisson ratio range is extended. The numerical example shows that it can be set to 0.18 ~ 0.34. Li et al.[32] proposed a local approximate strain tensor representation method. Through this representation method, the relative displacement local strain calculation between the material points is completed. Meanwhile, it was proved that the bond stretching amount is independent of the rigid rotation tensor. As shown in Fig. 6, compared with FEM, the model established by Li et al.[32] has wide validity, and the numerical example shows that it can be set to −1 ~ 0.5.

Zhu et al.[20] proposed a micro-elastic modulus based on local shear strain. By considering the rotation effect of bonds, PD theory was reformulated and enriched to describe the shear deformation of solids more precisely, as shown in Fig. 7. The proposed theory could reduce the restriction of Poisson's ratio effectively. Meanwhile, the effective elastic tensor was derived under the thermodynamic framework. On this basis, Liu et al.[33] proposed an improved method. By introducing an influencing factor related to the bond length, it can not only break through the Poisson ratio limitation, but also the calculation results are more consistent with the continuum mechanics as well.

Fan et al.[34] proposed a PD formulation based on two-dimensional micro-potential energy. By decomposing the bond length strain into two parts: volume strain and deviatoric strain, it can finally be applied to the solution of plane stress and strain problems. The numerical examples show that the Poisson ratio can be set to an arbitrary value. As shown in Fig. 8, the calculated results of the two displacement components are in good agreement with those of FEM.

Chen et al.[35] proposed a BB-PD correspondence model. By referring to the deformation gradient polar decomposition process in continuous medium mechanics, the deformation of the bond is divided into a rotating part and an elongated part. Meanwhile, the deformation gradient of the BB-PD was defined. Finally, the force density calculation completed by the physical equation is used to obtain the stress. The numerical example shows that it can take any value. Zhou et al.[36] proposed a micropolar BB-PD model. The governing equation of the material point system was derived from the minimum potential energy principle, and the non-local stiffness matrix was constructed. Finally, it can not only break through the Poisson ratio limitation but also provide a better solution for analyzing static/quasi-static problems.

(2) Bond rotation scheme

The bond rotation scheme refers to considering the axial action of the material point pair and the bending moment caused by the material point rotation, thus breaking through the Poisson ratio limitation. Zhou et al.[37] proposed a BB-PD model based on the material point rotation effect. By introducing the rotation micro potential into the PD, the coupling between the PD non-local properties and the material point rotation is finally realized. The novel conjugated bond linear elastic model incorporates two micro-elastic constants, which can overcome

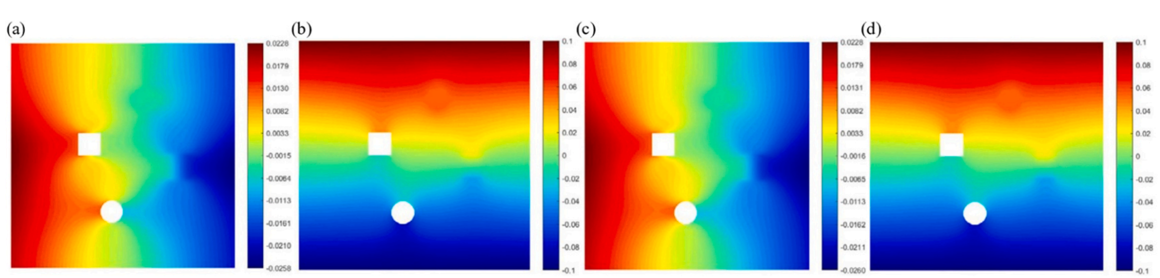

Fig. 6. Distribution of the displacement components with Poisson ratio is 0.2[32]: (a) *X*-direction by FEM; (b) *Y*-direction by FEM; (c) *X*-direction by this new model [32]; and (d) *Y*-direction by this new model[32].

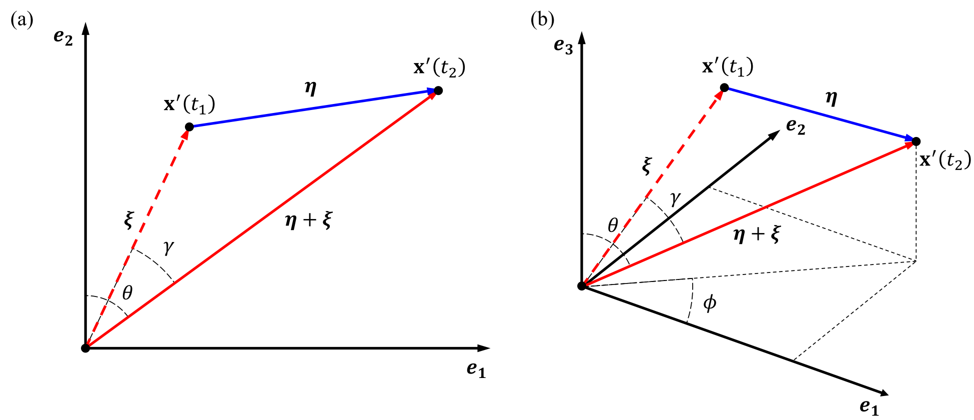

Fig. 7. Illustration of bond rotation[20]: (a) is two-dimensional case in polar coordinate; and (b) is three-dimensional case in sphere coordinate.

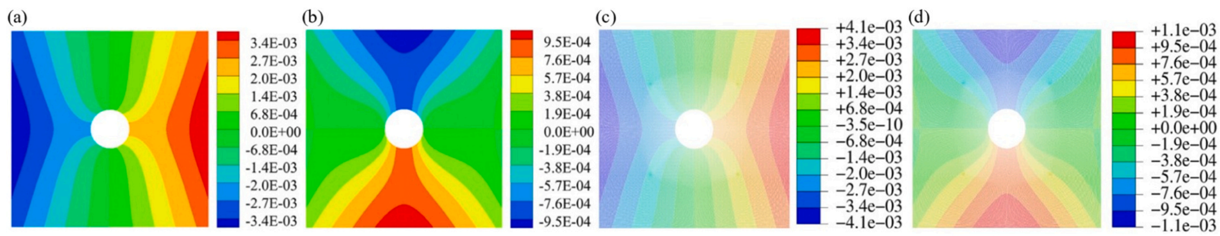

Fig. 8. Distribution of the displacement components[34]: (a) *X*-direction by this new model[34]; (b) *Y*-direction by this new model[34]; (c) *X*-direction by FEM; and (d) *Y*-direction by FEM.

the problem of Poisson's ratio in the standard BB-PD, and the multiple bond forces of this model are shown in Fig. 9.

Zheng et al.[38] proposed an improved BB-PD model. By calculating the axial and transverse force density of the bond, and adding the material point rotation angle calculation, the numerical example shows that the Poisson ratio can be set to 0.1 ∼ 0.33. Gu et al.[39] proposed a modified double-micro modulus conjugated BB-PD model. By establishing the relationship between the force density and the relative normal stretching, and considering the rotation angle of the conjugated bond, the stretching-rotation spring model was completed. As shown in Fig. 10, the correctness and applicability of the proposed model in solid deformation analysis are verified by comparison with FEM. The numerical example shows that the Poisson ratio can be set to 0.23.

It should be noted that a method that considers the conjugate bond rotation angle is an effective means of breaking through the Poisson ratio limitation. It has been increasingly concerned. For example, Zhou et al.[40,41] proposed a conjugate shear bond model and an improved conjugate bond linear elasticity model, both of which were applied to the BB-PD framework; Wang et al.[42] proposed a BB-PD model based on three-dimensional conjugate bonds, and the numerical examples show that the Poisson ratio can be set to 0.15 ∼ 0.4.

(3) New bond type scheme

The new bond type scheme refers to treating 'bonds' as different types and using different computational methods to reflect deformation behaviors. Such schemes generally have the advantages of simple implementation and clear principles.

Li et al.[19] proposed an improved BB-PD model by introducing normal bonds, tangential bonds, and pressure bonds. It finally alleviated the Poisson ratio limitation. Zhou et al.[43] proposed a two-dimensional non-local lattice BB-PD model. By considering the effects of normal and tangential bonds, the Poisson ratio limitation in the classical lattice model is finally alleviated. Meanwhile, in this model, multiple neighboring lattices are utilized to solve the elastic problem and the crack path preference problems, as shown in Fig. 11. Hu et al.[44] proposed a bond-type division method in the horizon. By dividing the bond type into normal bond and tangential bond, the force density function characterized by engineering constants was derived:

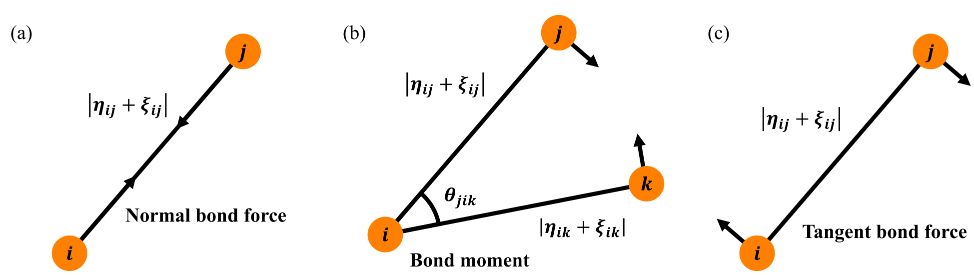

Fig. 9. Schematics of normal bond force, bond moment, and tangent bond force[37]: (a) is normal bond force; (b) is bond moment; and (c) is tangent bond force.

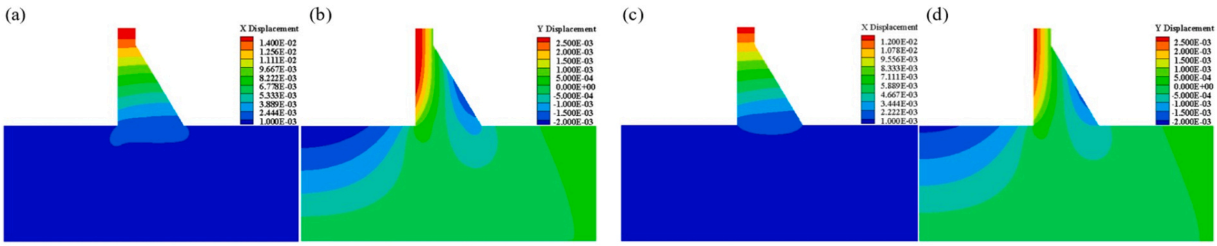

Fig. 10. Distribution of the displacement components[39]: (a) X-direction by BB-PD model[39], (b) Y-direction by BB-PD model[39], (c) X-direction by FEM; and (d) Y-direction by FEM.

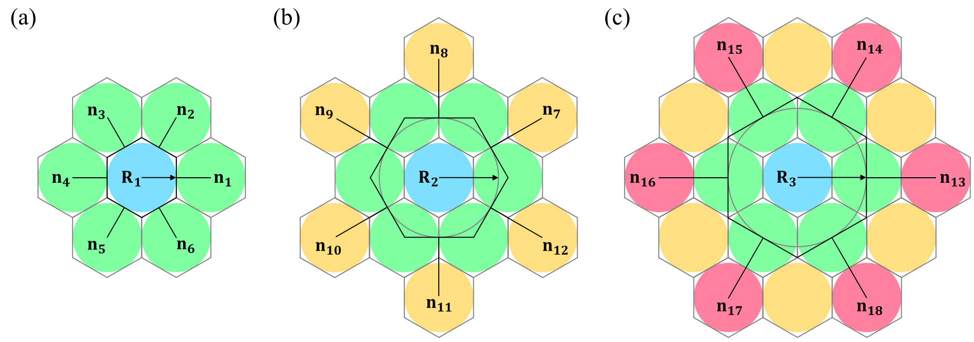

Fig. 11. The unit lattices of the non-local lattice BB-PD model[43]: (a) is 1st-nearest-neighbor; (b) is 2nd-nearest-neighbor; and (c) is 3rd-nearest-neighbor.

$$
\begin{cases}
f^{H}_{(i)(j)x} = \begin{bmatrix} c_{11} & c_{12} \end{bmatrix} \begin{bmatrix} s^{H}_{L} \\ s^{H}_{T} \end{bmatrix} \\
f^{H}_{(i)(j)y} = 0 \\
f^{V}_{(i)(j)x} = 0 \\
f^{V}_{(i)(j)y} = \begin{bmatrix} c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} s^{V}_{T} \\ s^{V}_{L} \end{bmatrix} \\
f^{S}_{(i)(j)x} = \dfrac{\left| x_{(j)} - x_{(i)} \right|}{\Delta x} c_{33}\gamma \\
f^{S}_{(i)(j)y} = \dfrac{\left| y_{(j)} - y_{(i)} \right|}{\Delta y} c_{33}\gamma
\end{cases}
\tag{18}
$$

where f is force density component; c_{11}, c_{12}, c_{21}, c_{22}, and c_{33} are PD parameters; Δx or $\Delta y$is discrete interval; $[\cdot]$ and γ is bond strain. Finally, the method can realize the numerical calculation of longitudinal stretching, transverse stretching, and the angle of two material points. The numerical example shows that the Poisson ratio can be set to any value.

(4) Multi-parameter description scheme

In general, conventional BB-PD converts all bond deformation into normal action, so that the Poisson ratio can only be taken to a specific value. The multi-parameter description scheme refers to the transformation of the bond deformation into two parts, normal and tangential, and calculates them separately. As shown in Fig. 12, Huang et al. [45] proposed a single-bond two-parameter PD model. It constructs a mathematical expression for the two-parameter (normal and tangential stiffness) force density, by the decomposition of the relative displacement vector in the normal and tangential directions. Finally, this model can not only inherit the traditional BB-PD simplicity and stability but also break through the Poisson ratio limitation. The numerical example shows that the Poisson's ratio can be set to $1 \sim 1/4$ under three-dimensional conditions and plane strain conditions, and $1 \sim 1/3$ under plane stress conditions. In addition, compared with RFPA, the current model also can capture the crack path, as shown in Fig. 13. Meanwhile, Huang et al. carried out parallel double crack extension simulation tests[46] and brittle material damage simulation tests under impact loading[47] by this model, both of which obtained better simulation results.

Li et al.[48] proposed an improved single-bond two-parameter PD

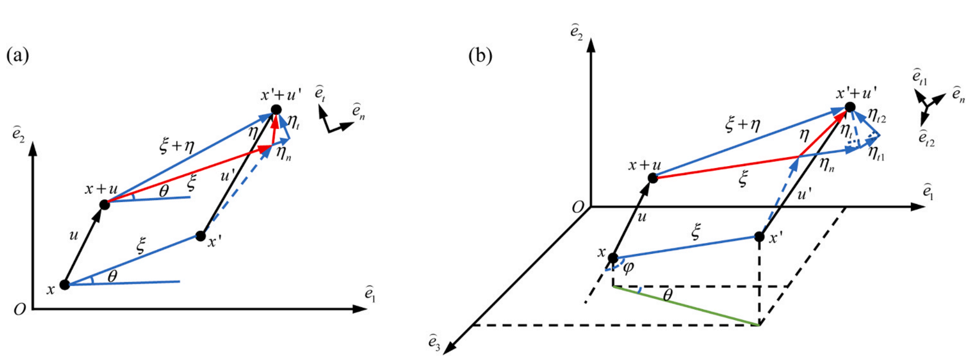

Fig. 12. Single-bond two-parameter BB-PD model[45]: (a) is a two-dimensional situation, and (b) is a three-dimensional situation.

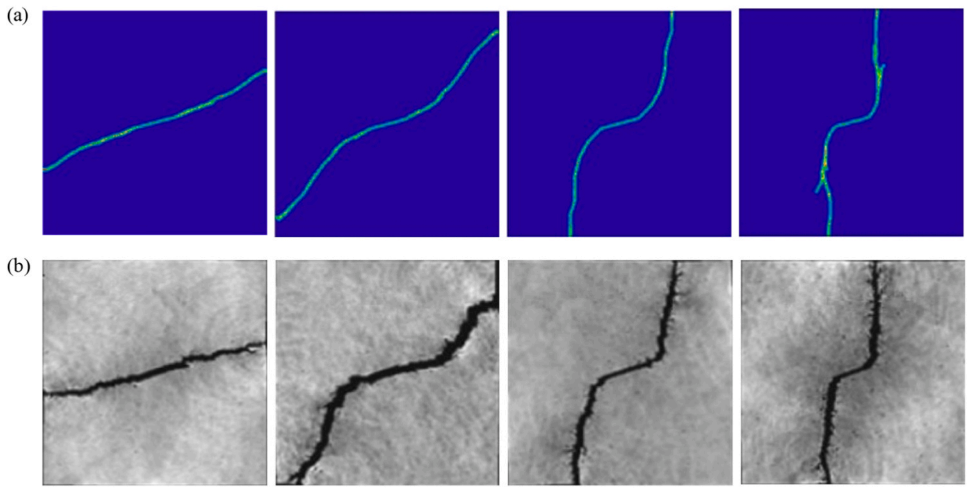

Fig. 13. The numerical simulation of crack path[45]: (a) single-bond two-parameter PD model[45]; and (b) RFPA model by Yang et al. [51].

model. The expressions of normal and tangential stiffness were characterized by elastic modulus and Poisson ratio. Meanwhile, a continuous function considering the internal length effect of long-range force was introduced. Finally, as shown in Fig. 14, by comparing crack patterns with many classical models, the current model can be used to simulate the linear and non-linear mechanical behavior of quasi-brittle materials.

In addition, by treating the 'bond' as a different mathematical or mechanical model, more parameters can also be introduced to describe the mechanical behavior of the 'bond'. For example, Gerstle et al.[49] proposed a micropolar PD model, which considers the 'micro-truss' between the material points as a 'micro-beam'. Chen et al.[50] proposed an improved orthotropic unidirectional plate BB-PD beam model. They have effectively alleviated the limitation.

It should be noted that the SB-PD proposed by Dr. Silling in 2007 considers the combined action of all bonds about the two material points [49]. It enhances the description of the intrinsic properties so that the SB-PD does not involve the problem of Poisson ratio limitation. Although the combined action can distinguish between the calculation of volumetric strain and deviator strain, it undoubtedly increases the various model complexity, resulting in a significant increase in computational effort and a reduction in numerical stability.

In summary, although the Poisson ratio problem has limited the development of the BB-PD applicability for a certain period, it has been effectively alleviated with related research. Assuming that acceleration techniques (e.g., algorithm optimization and parallel computing, etc.) are not considered, the BB-PD has the advantage of lower constitutive model complexity, computational efficiency, and numerical stability than the SB-PD. Especially, the non-ordinary state PD with the hourglass problem. The breakthrough of Poisson ratio limitation will make the BB-PD not only have the advantage of numerical calculation but also no longer be limited by the application scope.

3.2. *Various fracture criteria*

In 1920, Griffith was the first to introduce the concept of brittle materials fracture mechanics from the glass fracture research[55]. Since then, to investigate and elucidate the fracture mechanism in greater depth, various methods for predicting or judging the fracture initiation-expansion process have been developed. Many laboratory test results have shown that materials with small cracks can usually show strong resistance to fracture[56]. However, when classical continuum mechanics is used to reproduce this phenomenon, it often cannot reflect the crack size effect, i.e., contradictory results are obtained.

In general, the classical numerical methods based on the continuum mechanics theory mostly use pre-processing methods (e.g., prefabricated cracks or specified crack propagation directions, etc.) to simulate the crack initiation-expansion process. This artificial assumption and intervention will inevitably bring errors to the numerical simulation results. However, at the beginning of the derivation of PD theory, fracture characterizing function in scalar form has been introduced. Meanwhile, there is no need for any external auxiliary crack propagation method. It means that PD can simulate the crack's natural initiation-expansion process. In addition, the PD core equation also considers the calculation of local damage (it refers to the ratio of the disappearance interaction in the horizon to the total number of original interactions). This is used to reflect the degree of bond fracture in the horizon, as shown in Fig. 15. These characteristics give PD great advantages in dealing with material fracture and damage. Related research work is also more prominent[57]. For example, homogeneous/heterogeneous materials dynamic fracture[58–60], membrane and fiber structure damage and fracture[16,61], polycrystalline[62,63] and quenched glass brittle fracture[64], composite materials damage and failure analysis[65,66] and rock-like materials stability.[49,67,68]

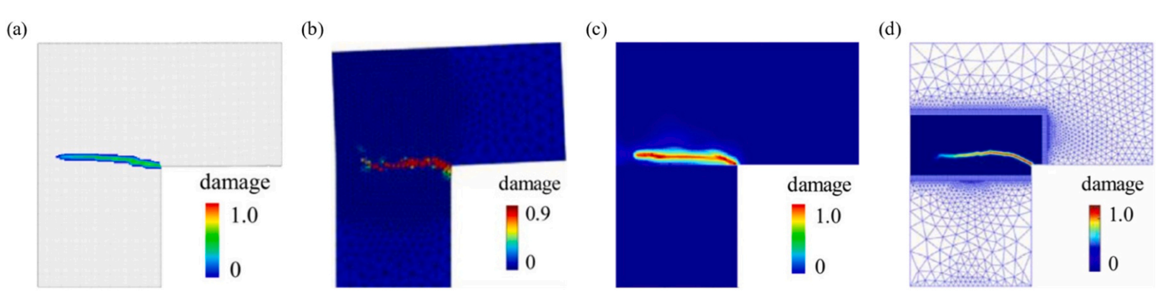

Fig. 14. Crack patterns of concrete L-specimen panel[48]: (a) the current model by Li et al.[48]; (b) the cohesive-frictional model by Le et al.[52]; (c) the new bond damage model by Tong et al.[53]; and (d) the cohesive zone PD model by Yang et al.[54].

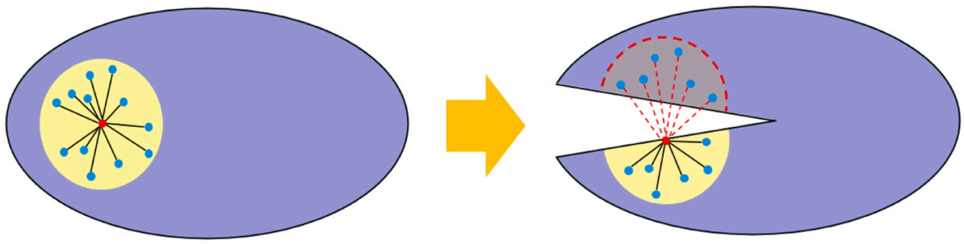

Fig. 15. Local damage of peridynamics (about 50 %)[57].

The PD theoretical framework is different from the traditional numerical simulation method. It is no longer limited by the linear elastic fracture mechanics theory. It is even more suitable for simulating the initiation-extension-bifurcation-intersection process of several cracks. However, in some special problems (e.g., hydraulic fracturing, etc.), it is debatable whether the bond fracture way needs to be improved. This is because the classical PD uses the bond critical elongation (it is similar to the maximum linear strain criterion in material mechanics) or the bond critical energy density to determine whether the bond satisfies the fracture condition. In this way, facing the problem that the macroscopic scale is characterized by shear failure or mixed-mode failure, whether it is reasonable to use the tensile failure criterion to simulate at the microscopic scale has become a question. It should be noted that for actual engineering, shear failure is often universal, abrupt, and catastrophic. Correctly capturing shear failure modes in different conditions is an essential test for any numerical model[56,69]. Moreover, the bond critical elongation and critical energy density are both functions of the horizon. They increase with mesh refinement. This will exhibit a strong mesh dependence.

At present, to deal with different fracture situations in actual engineering, various improved bond-related fracture criteria or various classical fracture criteria have been proposed or introduced into the PD. This is also the continuous improvement process of the PD theory.

(1) Various improved bond-related fracture criteria

The PD reflects the mechanical behavior and information exchange between the material points, by the concept of bond. The loss of connections between the material points by the failure of the bonds. This means that improved bond-related fracture criteria can not only satisfy the judgment of more fracture modes but also minimize the changes in the PD. In other words, they can maintain its original simplicity and stability.

Wang et al.[42] proposed a fracture criterion based on the bond critical stretch rate and critical shear energy density. By applying different fracture criteria to different fracture modes (mode-I for tensile fracture; mode-II for shear fracture; mode-III for tearing fracture), the numerical examples show that this model can predict the complex crack propagation in three-dimensional cases. Huang et al.[46] proposed a fracture criterion based on the bond critical strain energy density. By solving the strain energy density expression characterized by PD and classical continuum mechanics jointly, a fracture criterion suitable for the single-bond two-parameter BB-PD model was obtained. The numerical examples show that the fracture criterion greatly enriches the BB-PD theoretical model. This theoretical model has a good ability to capture the crack expansion path, as shown in Fig. 16. Yu et al.[70] proposed a fracture criterion based on the bond energy, and reformulated the mechanical behavior by realizing the tension-rotation-shear coupling effect. The numerical examples show that this criterion can describe the progressive failure process of quasi-brittle materials.

In addition, by considering various bond deformation effects, the analysis and judgment of various deformation processes and fracture modes can also be realized. Vito Diana[71] proposed two based-deformation micropolar PD fracture criteria. By deriving micro elastic energy functions about the micropolar non-local lattice, a generalized micropolar BB-PD model considering shear deformation was obtained. The numerical examples show that this model has a good simulation capability for dealing with linear/nonlinear shear deformation problems. Yan et al.[72] proposed a micropolar PD fracture criterion based on the shear effect. The shear effect was introduced into the PD by Timoshenko beam theory. Meanwhile, the PD parameters in two-dimensional and three-dimensional cases were derived by the energy equivalence principle. Finally, a micropolar PD model considering the shear effect was obtained. The numerical examples show that the model shows high prediction accuracy and numerical stability in simulating the various crack expansion processes.

It should be noted that by improving other bond correlations, it is also possible to simulate various effects that affect fracture behavior. For example, the interface effect, environmental effect, native defect, fatigue effect, and aging effect. Research in this area is in progress[73–75]. In 2005, Dr. Silling discovered that it is feasible to characterize the interfaces between composites using the different bond properties to reflect the interfacial mechanical behavior[60].

(2) Various classical fracture criteria

Compared with various improved bond-related fracture criteria, the theoretical derivation of the classical fracture criteria suitable for PD would be more difficult. This is because BB-PD or ordinary SB-PD doesn't involve traditional concepts, such as stress and strain. However, some scholars have realized the introduction of classical fracture criteria based on various theories. Qin et al.[76] proposed a BB-PD

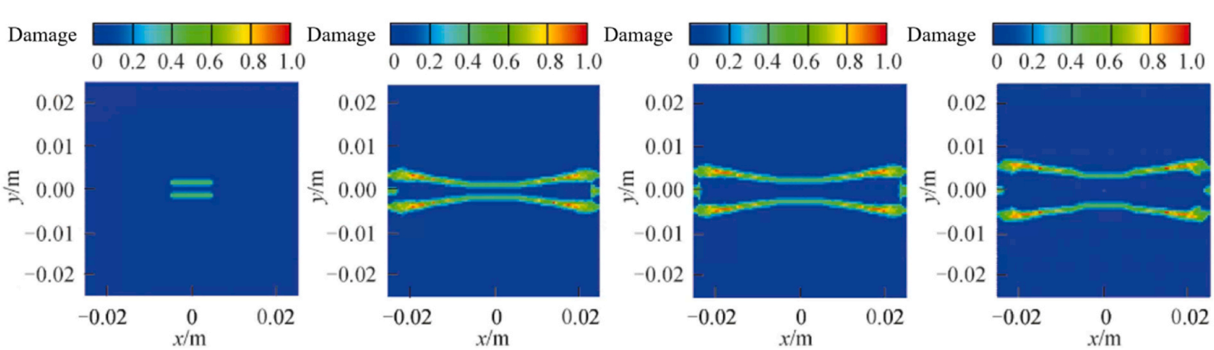

Fig. 16. The parallel double fracture extension calculation results of the reference[46].

model considering the Mohr-Coulomb criterion. The bond strain is calculated by the local coordinate system, the coordinate transformation matrix, and the bond displacement gradient:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi'_1 \\ \xi'_2 \end{bmatrix} \tag{19}$$

where $[\xi]$ is relative position vector and θ is coordinate angle. Based on the infinitesimal deformation assumption, the displacement gradient of the bond can be written as

$$G_d(\xi, t) = \begin{bmatrix} (x_2 - x_1)/x_1 & (y_2 - y_1)/x_1 \\ 0 & 0 \end{bmatrix} \tag{20}$$

Finally, a PD model suitable for simulating hydraulic fracturing was obtained. The numerical example shows that it can capture the crack morphology and propagation mode. It should be pointed out that compared with some classical fracture criteria (e.g., the maximum tensile stress criterion or maximum line strain criterion) for judging tensile failure, they believe that the bond critical elongation criterion also has good accuracy[76].

In 2007, Dr. Silling proposed the non-ordinary state-based theory. The deformation gradient tensor can introduce various classical constitutive relations or fracture criteria into the PD, thus it greatly enriches the PD theoretical system and application scope[21]. Cui et al. [77] proposed a non-ordinary SB-PD model considering the Mohr-Coulomb criterion. By using the Galerkin method to reconstruct the non-ordinary SB-PD, the deformation gradient solution is more generalized. Meanwhile, the introduction of the Mohr-Coulomb criterion is realized by using the physical quantities that have been included. Finally, a non-ordinary SB-PD model with high solution accuracy was obtained. The numerical examples show that the model has high prediction accuracy for the dynamic crack propagation paths, as shown in Fig. 17.

In summary, although the fracture criterion singularity has limited the further improvement of the PD theoretical system in a certain period, with the related research progress, various improved bond-related fracture criteria, and classical fracture criteria have been proposed and introduced into the PD. This has greatly enhanced the PD's ability to deal with various fracture modes. The improved bond-related fracture criteria can ensure the integrity of the PD theoretical framework, but the boundaries of their applicability are not clear. The classical fracture criteria can describe the materials' damage and fracture properties more easily, but only a few classical fracture criteria have been introduced. At present, there are difficulties in solving for Type II and mixed fractures. There are two main reasons. One is the difficulty in the correspondence decomposing between fracture type and bond deformation, and the other is that shear strain cannot be explicitly characterized by bond deformation[78,79]. To deal with more fracture situations in actual engineering, more fracture criteria and matching constitutive models are to be further studied[17].

3.3. Contact / Contact-friction model

With the introduction of various fracture criteria, the PD can better simulate and analyze the transformation process from continuous media to discontinuous media, e.g., crack initiation site, crack propagation speed, crack propagation path, crack development morphology, etc. This transformation process is often the precursor of structural instability or engineering disaster. Meanwhile, the corresponding PD theoretical system still needs to be improved to deal with the further evolution process after the discontinuous media formation. This evolution process can often provide important numerical references for safety design and disaster assessment[80,81]. The above two processes not only involve the two most concerned problems in disaster prevention and mitigation engineering but also are the necessary capabilities of PD as a reliable numerical simulation method for evaluating complex mechanical responses. This means that the discontinuous media behavior characterization algorithm is an indispensable improved part of the PD theoretical system.

The contact and friction behavior of discontinuous media are the two most common phenomena in the evolution process. In general, contact and friction forces may exist between two or more discontinuous media where contact occurs. At this moment, the contact and friction forces will be the main external forces on the discontinuous medium. However, Dr. Silling has mentioned that the Peridigm program (an open-source PD program developed by Sandia National Laboratories, USA) only responds to the discontinuous media contact behavior by simply applying a short-range repulsion[82]. This treatment does not have any physical significance and does not consider the friction coefficient, as shown in Fig. 18.

For this reason, various contact models or contact-friction models have been proposed and introduced into the PD for dealing with the

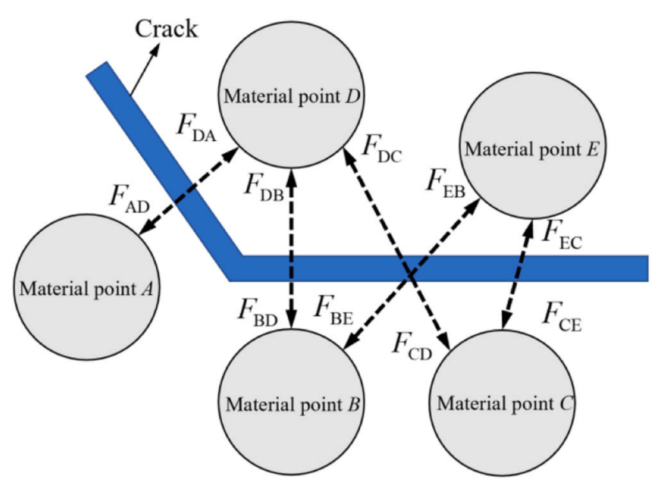

Fig. 18. Short-range repulsion diagram of Peridigm[82].

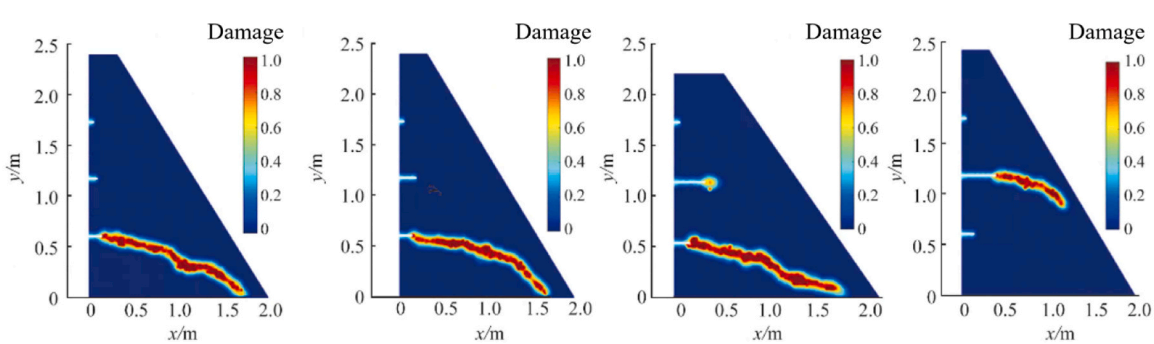

Fig. 17. Results of crack extension[77] in a dam containing multiple cracks.

further evolution process after the formation of discontinuous media.

(1) Contact models

Efficient and accurate contact models have always been the goal pursued by scholars, and various classical models have gradually emerged. For example, the Hertz model, the Spring-damping model[83], and the Nonlinear-damping model[84]. In the contact method based on material point/particle, the Nonlinear-damping contact model can not only overcome the shortcomings of the Spring-damping model but also retain the advantages of the Hertz model. From the implementation possibility, PD as a meshless method, the applicable contact model is rare. Dr. Silling[85] originally built only the Rigid Impact Contact Model and the Deformable Impact Contact Model, but the only function of the two models is to avoid discontinuous media interpenetration.

Since then, Timon et al.[86] proposed an improved Nonlinear-damping contact model based on the bucket sorting method. By treating the damaged material points as boundary material points and breaking through the constant horizon limitation, a double horizon PD model considering the material points' contact effect was finally obtained. The numerical examples show that this model can consider the contact effect caused by crack closure in the same area or different areas at the same time. LEE et al.[87] proposed a contact model suitable for the interaction between PD and FEM. By deriving and implementing the mapping contact algorithm from the material point to the material surface, and introducing the displacement constraint penalty function, a PD model suitable for analyzing the material impact and fracture process was obtained. The numerical examples show that this model can well reproduce the steel plate ballistic perforation test.

It should be noted that the contact model proposed by Timon et al. [86] focuses on material point contact detection. The contact detection method is the first task in many contact models. This is because, on the one hand, the contact detection results correctness is the basis for the computational results of the whole model. The results usually determine whether the contact force calculation is triggered. At this moment, incorrect detection results will not only lead to incorrect algorithmic results but also cause the computation to stall or even the computational crash. On the other hand, the contact detection method efficiency will directly affect the computational program. This effect is very significant. In other words, researching more efficient and accurate contact detection methods is to calculate contact forces more quickly and accurately.

(2) Contact-friction models

In the crack initiation-expansion process, the contact pressure of the crack surface and the contact surface friction coefficient will affect the crack initiation depth and crack expansion paths. However, the contact-friction model of PD considering the above factors is very rare. Initially, Dr. Silling[88] only proposed a Bond-based Frictional Contact Model (BFCM) for BB-PD, but this model requires that the contact point friction force must be parallel to the bond direction. Then, Kamensky et al.[89, 90] summarized two other well-known contact-friction models: the Naive Frictional Contact Model (NFCM) and the Non-ordinary State--based Frictional Contact Model.

Since then, contact-friction models considering more influencing factors and mechanical effects gradually came out. As shown in Fig. 19, Qin et al.[76] proposed a contact-friction model based on the Coulomb friction law. By establishing the relationship between the normal contact force and the tangential friction force, and introducing the adjustment coefficient to ensure the calculation convergence, a PD model considering the contact effect and friction effect of the crack surface was obtained. Wang et al.[91] proposed a contact-friction model considering the sliding effect. By decomposing the relative position vector based on the BFCM and NFCM, a PD model suitable for simulating the fatigue crack initiation-expansion process in the contact area was obtained. The numerical example shows that the model can predict the crack wear degree well. As shown in Fig. 20, Lu et al.[92] proposed an improved two-dimensional PD contact-friction model. By considering the contact force as a short-range force and introducing a bond-tilt failure criterion, the relationship between the crack geometry and the bond fracture was established. The normal and friction contact forces are regarded as the external forces, thereby redescribing the PD equation of motion:

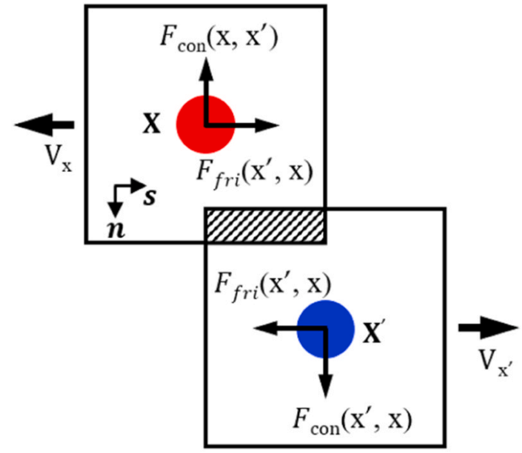

Fig. 19. Contact-friction model in reference[76].

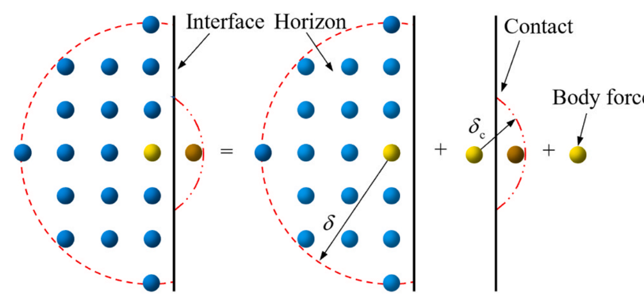

Fig. 20. Schematic diagram of force on a material point[92].

$$\begin{aligned}\rho\ddot{\mathbf{u}}\left(\mathbf{x},t\right) &= \int_H [\underline{T}(\mathbf{x},t)\langle\mathbf{x}''-\mathbf{x}\rangle - \underline{T}(\mathbf{x}'',t)\langle\mathbf{x}-\mathbf{x}''\rangle]\mathrm{d}V_{\mathbf{x}''} + \mathbf{b}\left(\mathbf{x},t\right)\\ &+\int_{H_c}\left[\underline{T}_n(\mathbf{x},t)\langle\mathbf{x}'-\mathbf{x}\rangle - \underline{T}_n(\mathbf{x}',t)\langle\mathbf{x}-\mathbf{x}'\rangle + \underline{T}_f(\mathbf{x},t)\langle\mathbf{x}'-\mathbf{x}\rangle - \underline{T}_f(\mathbf{x}',t)\langle\mathbf{x}-\mathbf{x}'\rangle\right]\mathrm{d}V_{\mathbf{x}'}\end{aligned} \tag{21}$$

where $\underline{T}_n$ is the normal contact force state (perpendicular to the contact surface) and $\underline{T}_f$ is the friction contact force state (parallel to the contact surface). Finally, a PD model suitable for characterizing the crack surface friction effect was obtained. The examples show that this model has the characteristics of realizing contact constraints without introducing any additional algorithm.

In summary, the simplified contact/contact-friction models constrain the application of PD in discontinuous media. With the improvement of the theoretical system, various contact / contact-friction models have been gradually proposed and introduced into the PD. However, the existing models still use distance to represent contact degrees (lacking physical meaning) and do not focus on contact detection efficiency, which has been a big challenge in compression-shear failure problems. Thus, a more efficient and accurate contact / contact-friction model is required to overcome the existing defects[93].

3.4. Constitutive / coupling model

For all kinds of actual engineering, safety design is a solution to fundamentally reduce economic losses and avoid casualties. From the standardization and rationality of engineering structures to the durability and safety of engineering materials, safety design usually faces many scientific problems[94,95]. Among them, whether the material properties (e.g., physical properties and mechanical properties, etc.) can show good working performance is a key issue. For example, as a special mining deep rock engineer, the mechanical properties of natural and engineering barriers of the high-level radioactive repository are complex. As the two most important aspects of its mechanical properties, strength characteristics, and deformation characteristics have received very extensive attention. Strength characteristics are usually described by strength criteria. For example, the various fracture criteria mentioned above. The deformation characteristics are usually described by the mapping relationship between the excitation and response of the material (constitutive relationship)[96,97]. The mathematical model or physical model used to reflect this macroscopic property is called the constitutive model. For example, the constitutive model is characterized by the relationship between the stress and strain tensor. The mature constitutive models can be used to quantitatively analyze the materials deformation law under different stress states.

With the increase in engineering demand and the deepening of related research, there have been nearly a hundred years of development history from the original constitutive models that can only describe the elastic deformation to various constitutive models that describe the complex mechanical properties[98]. Among them, the classical and widely recognized constitutive models mainly include the linear elastic model, ideal elastic-plastic model, ideal elastic-brittle model, cam-clay model, and so on. Although these classical models have their advantages/disadvantages and applicable conditions, they are indispensable for numerical simulation methods such as PD. For this reason, various classical constitutive models are constantly being introduced into the PD to deal with the various materials' different mechanical behaviors.

3.4.1. Classical constitutive model

While Dr. Silling and other scholars were deriving and proposing BB-PD and SB-PD, various constitutive models of linear isotropic materials (e.g., elastic, plastic, and visco-elastic models, etc.) suitable for PD had already been studied in some depth[49,99]. For example, Dr. Silling et al.[100] introduced microscopic elastic-brittle models into the BB-PD theoretical framework; Madenci and Oterkus[101] introduced plastic models into the ordinary SB-PD theoretical framework; Foster et al. [102] introduced visco-plastic models into an SB-PD theoretical framework; Sun and Sundararaghavan[103] introduced the crystal plasticity model into the PD theoretical framework first, and so on. For this reason, the constitutive model suitable for the PD has been developed, as shown in Fig. 21. Behera et al.[104] proposed a non-ordinary SB-PD model of creep deformation based on damage parameters. They simulated the creep crack growth in a compact tension specimen by this model. Dong et al.[105] proposed a PD creep-fatigue model to study creep-fatigue mechanical behaviors. They simulated mechanical responses by this model which are in good agreement with the experiment results conducted at high temperatures. It should be noted that BB-PD, as the original PD, also led the microscopic elastic-brittle model to become the first constitutive model suitable for brittle materials in the PD. Meanwhile, because the BB-PD does not involve stress and strain, the introduction of various classical constitutive models was greatly limited until the non-ordinary SB-PD theory was proposed.

Meanwhile, PD constitutive models considering more influencing factors and mechanical behavior are also making progress. Wu et al. [106] proposed a visco-elastic PD model considering the rate. By considering the damage evolution rate, viscous effect, and rate sensitivity, a PD model suitable for capturing the composite materials' dynamic response was obtained. Pathrikar et al.[107] proposed a visco-plastic PD model considering the damage. By introducing the plastic flow and material damage internal variables, and assuming the equilibrium state of the plastic flow and damage process, a PD model based on the energy equivalence was obtained. Chen et al.[108] proposed an elastic-brittle PD model considering the porous media damage. By introducing porosity as the initial damage and using randomly presetting cracks to detect and match porosity, a PD model suitable for calculating the wave propagation velocity of porous glass was obtained. Pashazad et al.[109] introduced an elastic-plastic models into SB-PD. By using the von Mises yield criterion to describe the plastic yield and describing the plastic model in the thermodynamic framework, a rate-independent PD model suitable for quasi-static problem analysis was obtained. Then, as shown in Fig. 22, the proposed model is in good

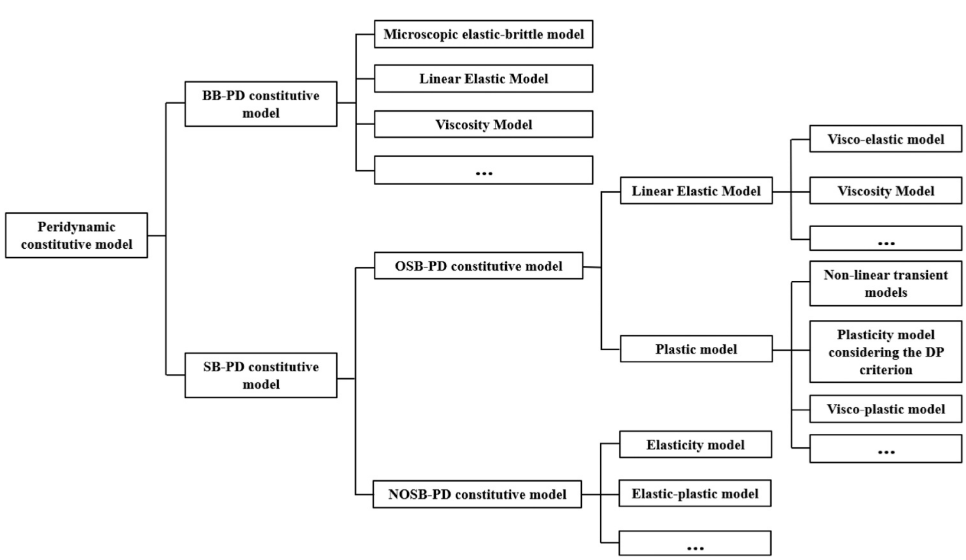

Fig. 21. Various constitutive models of peridynamics.

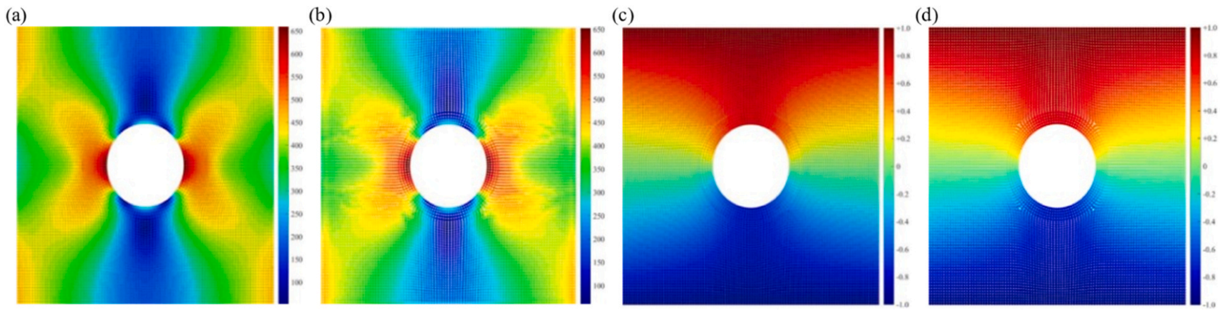

Fig. 22. Comparison of results[109]: (a) Mises stress by FEM; (b) Mises stress by the rate-independent PD model[109]; (c) *Y*-direction displacement by FEM, (d) *Y*-direction displacement by the rate-independent PD model[109].

agreement with FEM prediction results, which verifies the accuracy of the rate-independent PD model.

At present, the research on the BB-PD elastic-plastic constitutive model mainly focuses on the non-local constitutive model construction and the crack expansion processes qualitative study. The ability to quantify various post-peak physical quantities (e.g., critical loading, loading-displacement curve, crack expansion rate, etc.) is still lacking.

3.4.2. Multi-field coupling model

For various practical engineering, the engineering materials service environment usually has a different influence on their material properties. The interference will ultimately affect the maintenance of their working performance. For example, the load-bearing problems of saturated frozen soil and concrete structures in alpine areas, the cracking of concrete structures in arid/humid areas, the stability of surrounding rock in geothermal/petroleum resource extraction, the high-water pressure splitting problems of water conservancy dams, and stability of surrounding rock in high-level radioactive waste repositories, etc. It can be found that the service environment of engineering materials is usually very complex, often including various physical environments such as force, water (moisture, steam, and seepage), heat, electricity, magnetism, and chemistry.

At present, the multi-field dependence of various materials' physical properties[110–112] and mechanical properties[113–116] has been extensively studied. Take the most common problem of rock temperature dependence as an example[117,118]. Many experimental data show that although the rock under high-temperature environments is not subject to any external constraints, the expansion process between its internal minerals will also cause higher thermal stress. When the thermal stress exceeds the rock strength, the original crack expansions and the new crack generations will significantly affect the rock properties. For example, it aggravates the rock discontinuity[119]. It should be noted that with the improvement of relevant theories and experimental equipment, the basic understanding of the various materials' thermophysical behavior and thermomechanical properties has been significantly improved[111,120]. However, some materials' evolution behaviors under high-temperature conditions are not easy to track and evaluate. For example, the migration of various energy is a very rapid and highly complex process during crack expansion. It is not easily captured in the laboratory. This highlights the advantages of numerical simulation methods[121].

With the continuous deepening of research, the demand for multi-field coupling models or simulation software, which has stronger solving capabilities and higher calculation accuracy, is becoming more intense in various engineering[122]. At present, commercial software such as COMSOL, ANSYS, and ADINA have widely adopted or developed multi-field computing platforms. However, there are still obvious shortcomings when facing discontinuity problems such as media damage[123]. This means that the development of multi-field coupling models based on PD can provide beneficial tools for the study of multi-field coupling problems. The research progress of various PD multi-field coupling models is summarized below.

(1) Thermo-mechanics coupling model

The coupling of the temperature field and stress field is one of the most concerned multi-field problems in the PD. There are two main ways to establish the PD thermo-mechanics coupling model: the unidirectional and the bidirectional coupling[64]. The former only considers the temperature's influence on deformation. The bidirectional coupling mode considers the temperature influence on deformation and the deformation influence on thermal performance. Meanwhile, thermo-mechanics coupling models considering more influencing parameters and treatments have been proposed.

As shown in Fig. 23, Yang et al.[121] proposed an ordinary SB-PD fully coupled thermo-mechanics model. The SB-PD equation of motion for a linearized thermoelastic material is

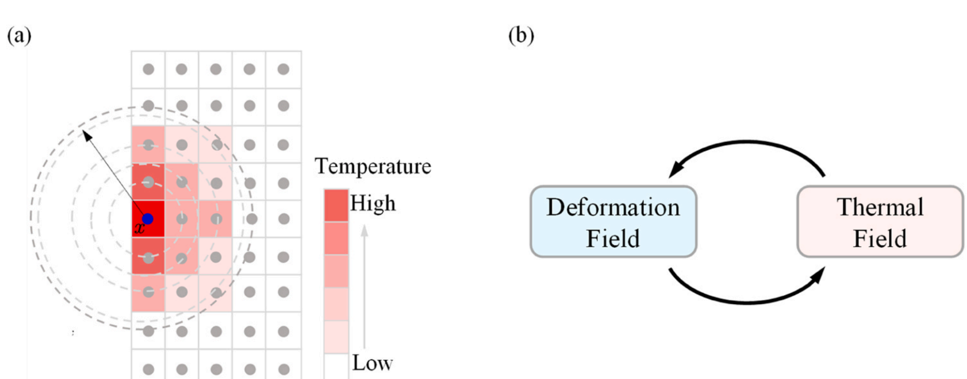

Fig. 23. An ordinary state-based peridynamics fully coupled thermo-mechanics model in the reference[121]: (a) is a thermal diffusion behavior diagram; and (b) is a two-field coupling schematic diagram.

$$\rho\ddot{\mathbf{u}}\left(\mathbf{x},t\right)=\int_H\left[\begin{array}{l}(\underline{T}(\mathbf{x},t)\langle\mathbf{x}'-\mathbf{x}\rangle-\underline{B}(\mathbf{x},t)\langle\mathbf{x}'-\mathbf{x}\rangle T)-\\(\underline{T}(\mathbf{x}',t)\langle\mathbf{x}-\mathbf{x}'\rangle-\underline{B}(\mathbf{x}',t)\langle\mathbf{x}-\mathbf{x}'\rangle T')\end{array}\right]\mathrm{d}V_{x'}+\mathbf{b}\left(\mathbf{x},t\right) \tag{22}$$

where, $\underline{T}(\mathbf{x},t)$ and $\underline{B}(\mathbf{x},t)$ are the non-local internal force states defined in the SB-PD, and the angle brackets, <>, indicate the vector on which the state field operates. Meanwhile, the thermal diffusion equation in the fully coupled thermomechanical PD model is provided as

$$\rho c_v\dot{T}\left(\mathbf{x},t\right)=\int_H[(\underline{h}[\mathbf{x},t]\langle\mathbf{x}'-\mathbf{x}\rangle-\underline{h}[\mathbf{x}',t]\langle\mathbf{x}-\mathbf{x}'\rangle)-\Theta_0\dot{e}\beta]\mathrm{d}V_{x'}+h_s\left(\mathbf{x},t\right) \tag{23}$$

where c_v is the specific heat capacity; $\dot{T}(\mathbf{x},t)$ is the time rate of the temperature change; $\underline{h}[\mathbf{x},t]$is the heat flow state; Θ_0 is the reference temperature; $\dot{e}$ is the time rate of the stretch extension change; and β is the local thermal modulus. By transferring the thermal damage to the material layer's corresponding position, and using a multi-layer calculation method to eliminate the crack change errors caused by the temperature gradient, a PD model suitable for rock materials' real-time temperature treatment was finally obtained. The numerical examples show that the model can accurately simulate the rock materials' deformation behavior and cracking process under high-temperature conditions. Wang et al.[124] proposed a BB-PD thermo-mechanics coupling model. By introducing the penalty function to realize the thermal crack random initiation, and using a multi-rate time integration scheme to overcome the time scale in thermal systems, a PD model suitable for simulating cyclic thermal fracture behavior was finally obtained. As shown in Fig. 24, compared with many classical methods, the numerical examples show that the BB-PD thermo-mechanics coupling model can accurately predict the thermal crack propagation mode, and the results are in good agreement with the existing results.

In addition, Wang et al.[125] also proposed an ordinary SB-PD weak thermo-mechanics coupling model. By using different methods to realize heat conduction and mechanical deformation, a PD model suitable for simulating the rock thermal fracture behavior under borehole heating conditions was finally obtained. The numerical examples show that the model can accurately capture the rock thermal deformation and fracture process. Pathrikar et al.[126] proposed a non-ordinary SB-PD fully coupled thermo-mechanics model. By deriving partial differential forms of the deformation-damage-temperature evolution control equations, and considering the combined effects of local heat generation and thermo-mechanics coupling at the crack tip, a PD model based on the entropy equivalence was obtained. The numerical example shows that the model can deal well with the discontinuous problems. For example, crack initiation and the high-order derivatives caused by the coupled partial differential equations.

Meanwhile, the PD is also suitable for reproduction phenomena such as thermal loading, heat conduction, and thermal diffusion. For example, Hu et al.[129] proposed a BB-PD and SB-PD heat conduction model for irregular modeling. By simulating the fracture process of three-dimensional nuclear fuel pellets under thermal loading, and analyzing the pellet cracking temperature, displacement, and damage distribution, the model's effectiveness in dealing with structural deformation and heat conduction was verified. He et al.[130] proposed an extended ordinary SB-PD thermal shock damage model. By introducing the thermal expansion coefficient correction term into the PD, and introducing the influence function characterizing the gradient change into the force state, the crack expansion process during the steel plate quenching process was accurately predicted. Wang et al.[131] proposed a PD non-Fourier heat conduction model. By introducing the two-phase lag concept into the SB-PD, and dealing with the compatibility of non-Fourier effect and non-local effect, the transient heat conduction from nano-scale to macro-scale can be better reflected.

(2) Fluid-solid coupling model

The traditional porous media elasticity theory cannot simulate discontinuous deformation problems, such as soil drying shrinkage cracking, and hydraulic splitting. Scholars developed the porous media elasticity theory under the PD framework to describe the liquid flow and solid deformation in porous media.

As shown in Fig. 25, Ouchi et al.[132,133] proposed a fluid-solid coupling model for heterogeneous poroelastic media. It is suitable for fluid-driven fractures. Through the pore flow equation including permeability coefficient and porosity, the fluid movement process in the fracture area is described, and the interaction of hydraulic fractures and single/multiple natural fractures is simulated.

Edmiston et al.[134] proposed a two-way fluid-solid coupling model for liquid flow and solid deformation. By incorporating the body force density into the PD motion equation, the local seepage equation and the BB-PD constitutive equation were jointly used. Finally, the characterization of material deformation by permeability coefficient was discussed. Oterkus et al.[135] also proposed a two-way fluid-solid coupling model for poroelastic media. By introducing the fluid pore pressure into the BB-PD motion equation, a non-local form of Darcy's seepage equation was established to explore the solid deformation. Finally, this model successfully reproduced the classic five-point confluence problem, one-dimensional/two-dimensional consolidation problem, and hydraulic fracturing. Meanwhile, the hydraulic fracturing results are in good agreement with those of ANSYS, as shown in Fig. 26.

(3) Mechanics-chemistry coupling model

In various storage/disposal engineering, the corrosive environment can easily cause pitting corrosion of engineering materials. It will lead to corrosive damage or cracking of materials or structures. The pitting corrosion can be regarded as the dissolution and diffusion problem of ions in solid/liquid dual-phase materials. At the beginning of the design of some storage/disposal engineering, safety plans need to be designed for certain special situations, such as nuclide migration in high-level radioactive waste disposal. These problems involve the coupling effect of the mechanical field and chemical field. Chen et al.[136] proposed a PD pitting activation and diffusion model and studied the impact of the surface passivation film on the structural pitting damage process. On this basis, Jafarzadeh et al.[137] further studied the passivation film and salt film formation model. De Meo and Oterkus[138] proposed a PD pitting damage model, as shown in Fig. 27. They efficiently simulated the two-dimensional material pitting damage and cracking problem by the implicit solving algorithm in Ansys. The new model allows for the reproduction of realistic pitting morphologies, the modeling of microstructural effects, and the reduction of computational cost.

(4) Water-mechanics-thermal, ion concentration-mechanics, thermo-electronics, mechanics-electronics, and thermo-mechanics-electronics coupling models

For electronic packaging materials or structures, there are also many brittle materials, such as ceramics and glass. However, the hot and humid environment often causes the electronic packaging structure to

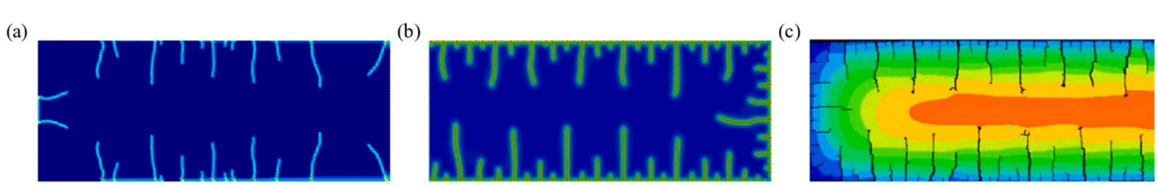

Fig. 24. Thermal cracking patterns[124]: (a) BB-PD thermo-mechanics coupling model[124]; (b) phase-field[127]; and (c) FEM[128].

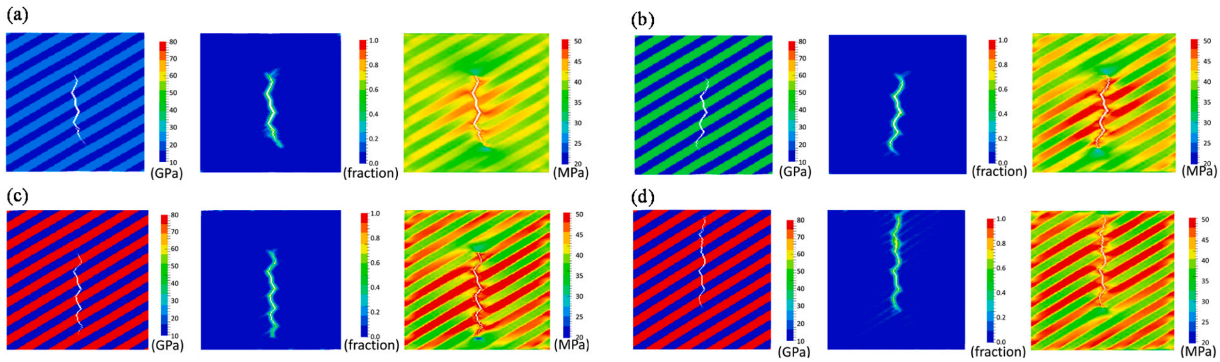

Fig. 25. Reservoir attribute distribution Schematic diagram in the reference[133].

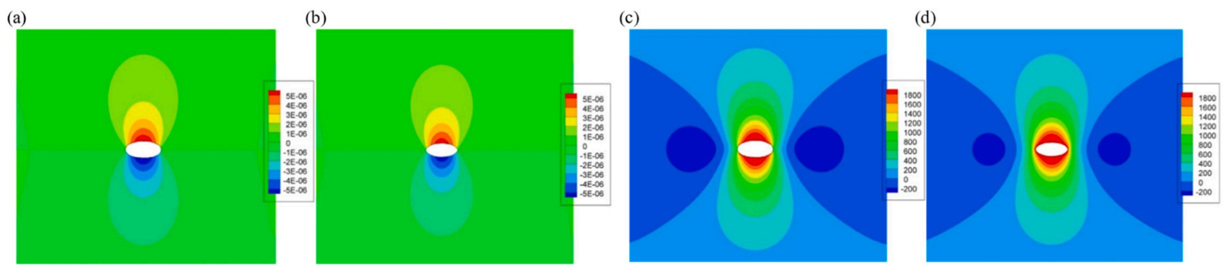

Fig. 26. Comparison of results[135]: (a) *Y*-direction displacement by the fluid-solid coupling model[135]; (b) *Y*-direction displacement by ANSYS; (c) fluid pore pressure by the fluid-solid coupling model[135]; and (d) fluid pore pressure by ANSYS.

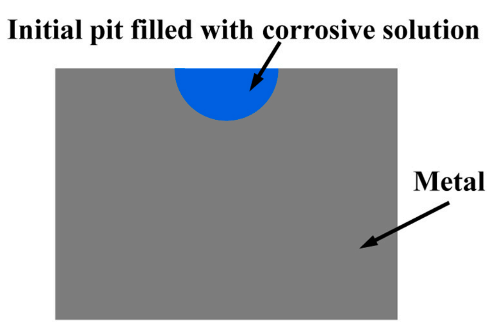

Fig. 27. The two-dimensional pitting corrosion model, initial pit (blue color), and metal (grey color)[138].

absorb water. This can cause deformation or damage to the structure.

Oterkus et al.[139] proposed a water-mechanics-thermo coupling model of BB-PD. By constructing a BB-PD constitutive force function that includes deformation, temperature, humidity, and vapor pressure, the calculation methods of the heat conduction model, moisture concentration diffusion model, and vapor pressure were discussed. Wang et al.[140] proposed an ion concentration-mechanics coupling model of BB-PD. By the BB-PD differential operator, the lithium-ion concentration equation was transformed into the corresponding non-local integral form. The influence of lithium-ion concentration was considered. As shown in Fig. 28, compared with FEM, the concentration and displacement results are similar to those of FEM.

It should be noted that for electronic packaging materials or dielectric materials, various electrical coupling models based on PD are also making progress. Zhang et al.[141] proposed a PD thermo-electronics coupling model based on Voronoi cells. By considering the Seebeck effect, Peltier effect, and Thomson effect, the thermoelectric conversion efficiency of three-dimensional thermoelectric devices was analyzed. Prakash and Seidel[142,143] proposed a BB-PD mechanics-electronics coupling model. By changing the conductivity and piezoresistive coefficient, the unidirectional influence of material deformation on electron transition and potential distribution was analyzed. Wildman and Gazonas[144,145] proposed a PD thermo-mechanics-electronics coupling model suitable for dielectric materials. By introducing the Joule heating effect in the PD constitutive force function, the Lorentz force was incorporated into the unified representation. Finally, the crack formation mode and brittle failure process of the dielectric material square plate were simulated.

In summary, with various constitutive models and coupling models being proposed and introduced into the PD, its theoretical system is

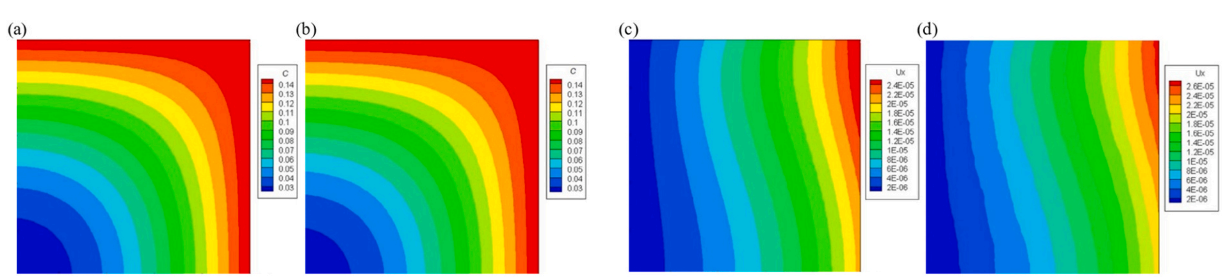

Fig. 28. Comparison of results[140]: (a) lithium-ion concentration distribution by FEM; (b) lithium-ion concentration distribution by the BB-PD model[140]; (c) *X*-direction displacement by FEM; and (d) *X*-direction displacement by the BB-PD model[140].

being continuously improved. This makes PD, as an emerging and reliable numerical simulation method, show great application prospects and development potential. However, there are still significant development difficulties. The first is the connection between PD and classical continuum mechanics is not close enough, and most of the existing coupling models are based on BB-PD. Non-ordinary SB-PD can only alleviate this dilemma. The other one is coupling models that involve interdisciplinary collaboration, e.g., thermodynamics, chemistry, electronics, and so on. The coupling model can be divided into a multi-field single-medium model or a single-field multi-medium model in PD. The multi-field multi-medium rock coupling model is still a great theoretical challenge. Because complex coupling models need to fully consider the compatibility (coupling mode) between multiple disciplines. However, multiple control equations will undoubtedly increase the stability, simulation scale, and computational efficiency of the solution system, i.e., increase the difficulty of model construction. Now, in computational mechanics, compared with multi-media, it is a more recognized way to prioritize the construction of multi-physical field coupling effects, which will be an effective way for PD to solve current problems. It is worth mentioning that in addition to rock engineering, PD is showing great application potential and prospects in multiple disciplines such as biomechanics, medical engineering, aerospace engineering, and marine engineering, e.g., numerical modeling of artificial organs, muscle motion analysis models, wing ceramic heat dissipation analysis and submarine icebreaking process simulation.

The above problems bring difficulties to the compatibility or introduction of many constitutive models and coupled models. However, it is not difficult to find, they also pointed out the direction of PD improvement.

3.5. Computing accuracy

In general, in multi-scale and multi-field coupling conditions, the numerical method can consider and analyze the environmental factors variation and the structure's mechanical behavior. While reducing the number of laboratory and field tests to save costs, it can equivalently and effectively shorten the analysis cycle, and finally improve the actual engineering safety design success rate. This is undoubtedly the unique advantage of the numerical simulation method and its development goal [131,146,147]. As an emerging numerical simulation method, PD calculation accuracy has always attracted much attention. This has gradually become an evaluation standard for the PD application value and prospect. It should be noted that in a finite material body, there is a difference between the calculation results of PD and the classical analytical solution of linear elastic uniform deformation. There are three main reasons for this difference. They are the numerical quadrature accuracy, the non-local boundary conditions realization, and the PD surface effect. Among them, the influence of the PD surface effect is the most significant, and corresponding improved algorithms have been proposed.

(1) The numerical quadrature accuracy

The numerical quadrature accuracy is mainly related to the mesh volume calculation way. It is mainly affected by two aspects: mesh division and volume contribution. Firstly, as a meshless method, the PD numerical implementation will inevitably involve spatial discretization [148]:

$$\rho \ddot{\mathbf{u}}(\mathbf{x}_i, t) = \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \mathbf{f}\left(\mathbf{u}(\mathbf{x}_i, t), \mathbf{u}\left(\mathbf{x}_j', t\right), \mathbf{x}_i, \mathbf{x}_j', t\right) v_{c\left(\mathbf{x}_j'\right)} V_{\left(\mathbf{x}_j'\right)} + \mathbf{b}(\mathbf{x}_i, t) \tag{24}$$

where $\mathbf{x}_i$ is the i-th central material point; N_i is the number of central material points or horizon; N_j is the number of family material points, which belong to the i-th central material point; $\mathbf{x}_j'$ is the j-th family material point of the i-th central material point; and $V_{\left(\mathbf{x}_j'\right)}$ is the volume

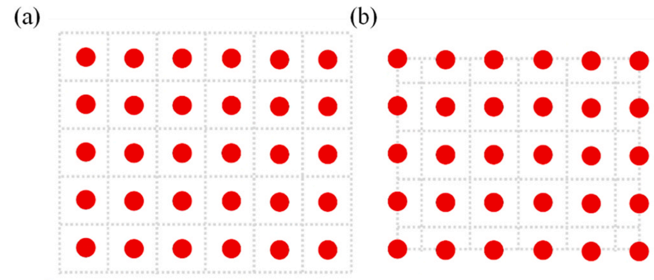

Fig. 29. Spatial discretization methods of peridynamics[148]: (a) is the first method.

of a family material point. It can be found that this will lead to the error caused by the discretization introduction. Currently, as shown in Fig. 29, there are two main methods for PD spatial discretization. The advantage of the first method[148] is that the square mesh edges can coincide with the computational model, any material point volumes in the computational model are the same, and the numerical modeling implementation is relatively simple. The disadvantage is no material points on the boundary of the computational model. This makes it difficult to apply the boundary conditions. However, this discretization method is often the preferred method for involving material damage and fracture problems. The advantage of the second method[16] is the aligned mesh discretization method. The volume of the material points at the boundary is 1/2 of the central area, and the volume of the material points at the corner is 1/4 of the central area. There are material points on the boundary of the computational model. The disadvantage is that the material points around are no longer located in the mesh center. The numerical quadrature accuracy of these positions will be affected, but the existing correction algorithms can make up for this disadvantage [148].

For the second aspect, PD follows the idea of solving the volume integral in the horizon. To avoid incompleteness of the discrete mesh volume at the edge of the horizon, various volume correction algorithms have been proposed. The volume correction factor is the most commonly used method. However, the way only mitigates the volume loss to some extent, i.e., it is also a numerical approximation treatment way. As shown in Fig. 30, take the blue material points No. 1–5, as examples, the volume correction factor of both material points No. 1 and No.5 is 0.5, but the volume contributed by these two material points should both be less than the 1/2 of the discrete mesh volume. The No.3 is about 0.67. It is closer to the intersection area proportion obtained by the Area Compensation Method. The No.2 and No.4 are 0, but there is a volume contributed by these two material points. This is because they are not considered family material points. It should be noted that this error can be reduced by increasing the discretization degree, but this problem will remain[149].

In addition, the calculation parameters (e.g., elastic modulus, etc.) also produce some errors. This is due to the numerical approximation of the discretized PD model. Existing studies have shown that, compared to

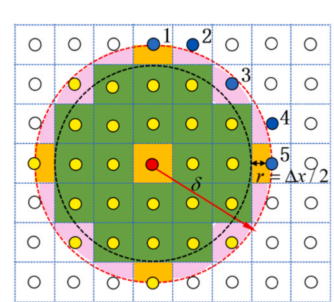

Fig. 30. Volume correction factor[149]and (b) is the second method.

the PD surface effects, by investigating the PD model discretization convergence degree, it is found that the errors caused by spatial discretization and numerical approximation are very small[148]. Meanwhile, in the simulation process, using a higher ratio of the horizon to the discrete interval will have a numerical quadrature accuracy closer to the original continuous formula[150–153].

(2) The non-local boundary conditions

In theory, the boundary condition essence is the description (or definition, designation, simplification, etc.) of the forces acting on the object by the surrounding objects. In general, the boundary condition setting of a given displacement is completed by the geometric equation, the boundary condition setting of a given surface force is completed by the surface equilibrium condition, and the mixed boundary condition setting is completed by considering the above two ways. Since the PD core equation adopts the spatial integral form, the application of the boundary conditions is also different from that of the traditional classical continuum mechanics. The stress and force density cannot be used as the boundary conditions of the PD model.

As shown in Fig. 31, Dr. Silling's team[21], through several sets of numerical experiments, suggested that a virtual layer with a thickness of δ (i.e., this thickness is equal to the radius of the horizon) be placed outside the material boundary. Then the velocity or displacement is applied to the material points (virtual material points) in that virtual layer to complete the displacement boundary imposition. Meanwhile, the stress boundary condition is imposed by converting the surface forces into body forces (dividing by the cross-sectional area). The applications of these boundary conditions have certain reliability and advantages. For example, the application of stress boundary conditions conforms to the equivalence of the Saint-Venant principle and is easy to implement.

(3) The PD surface effect

In defining the PD computational parameters, it is necessary to satisfy the assumption that the material point has a complete horizon. However, As shown in Fig. 32, the horizons of material points 2 and 3 are not a complete circular domain. It should be noted that the incomplete horizon is common near the boundary. When the material point near the boundary uses the same computational parameters as the internal material point, the mechanical behavior of this material point is not the same as that of the internal material point. This is the PD surface effect.

The BB-PD and SB-PD will face this problem at the surface. Most of the material points on the surface show an under-stiffness mechanical behavior. This is because the number of material points in the family is relatively small. The PD surface effect may lead to obvious deviations in the material point displacement. The displacement inaccuracy usually directly affects the resulting inaccuracy, e.g., force density and energy. The existing research results show that if the same bond parameters are used for calculation, it will lead to the overall deviation of the material displacement calculation results. It should be noted that this effect is different from the boundary condition problem in the meshless method.

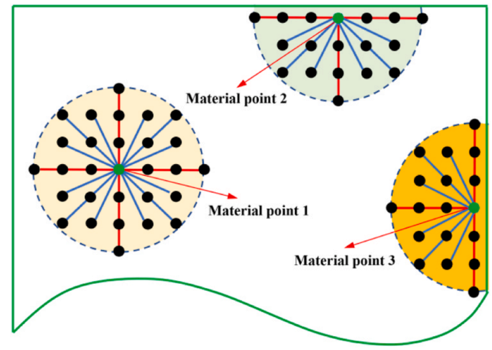

Fig. 32. Two kinds of material point horizons in peridynamics.

The meshless method usually uses the partial differential equation of traditional classical continuum mechanics. In other words, the constitutive model is independent of the material point position in these methods[154–159].

Initially, scholars tried to attenuate this surface effect by using a smaller horizon. However, the horizon size is directly determined by the material microstructure. It is not subject to human interference in some problems (e.g., homogenization in reference[160]). The smaller horizon is often accompanied by greater computational consumption. This eventually promoted the introduction of various surface effect correction methods. There are two main ideas for PD surface effect correction methods. The first is to modify the force density of material points near the surface so that the material point's mechanical behavior is similar to the internal. This method mainly includes the Volume Method[161], Force Density Method[162–164], Energy Density Method[163,164], Force Standardization Method[26], and Position Perception Method. The second is to try to restore the missing points on the surface so that each material point feels like the internal point. The Virtual Material Point Method[164–166] is the representative method of this idea.

Several methods mentioned above are mainly for BB-PD. By increasing the micromodule around the material point on the surface, the Volume Method finally makes the material point have the same strain energy density as the internal point under uniform deformation conditions. The Force Density Method divides the material points on the surface into regions (positive region and negative region) and characterizes the force density component. Meanwhile, the scale factor and scalar constant are introduced, and finally, the micromodule of the material points on the surface is close to the internal. The principle of the Energy Density Method is very similar to the Force Density Method, but the displacement difference generated by the Energy Density Method will be slightly larger. The Force Standardization Method calculates the restoring force required for a material point to move a certain distance from its equilibrium position. Then it compares this restoring force with the force required for an internal point under the same conditions. Finally, the mechanical behavior of the surface points is similar to that

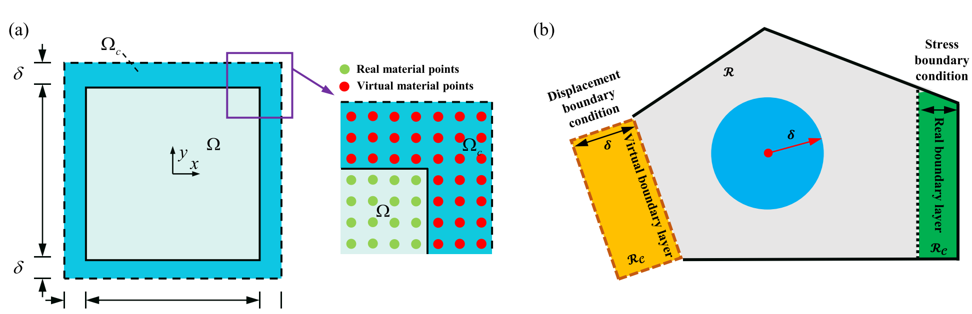

Fig. 31. The nonlocal boundary conditions of peridynamics[49]: (a) real and virtual material points in the reference; and (b). displacement and stress boundary condition.

internal ones. Different from these methods, the Position-aware Linear Solid Constitutive Model is not only a Position Perception Method but also an SB-PD model. By using the position perception influence function, this method finally realizes different calculation parameters of material points at material different positions. The existing research results show that the above methods can alleviate the PD surface effect. Facing different situations, the mitigation effect will be different, but they cannot eliminate the influence of the PD surface effect.

The Virtual Material Point Method adds virtual matter points around the material discretized system. Each real material point eventually has a complete horizon. In this case, the thickness of the virtual material point layer must be at least equal to δ. The virtual material points are forced to act on the real material points to construct the required non-local boundary conditions. The advantage of this correction method is that it does not require the PD core equations modification. This makes it applicable to almost all BB-PD and SB-PD models. The PD surface effect can be eliminated by compensating for the horizon of the surface point. The disadvantage is the dependence on the discretization system and the difficulty in compensating for virtual material points on the surface when faced with complex geometric shapes, as shown in Fig. 33.

In summary, compared with the traditional classical continuum mechanics, PD as a meshless method, cannot have a natural boundary, which makes the PD calculation accuracy not only dependent on the numerical quadrature accuracy but also limited by the material points on the surface. At this moment, there are many surface effect correction algorithms for alleviating the PD surface effect, each with its advantages and disadvantages. Although facing different situations, the surface effect correction algorithm can be selected to appropriately alleviate or eliminate the surface effect. However, the existing correction algorithms are not very universal and still need to be developed.

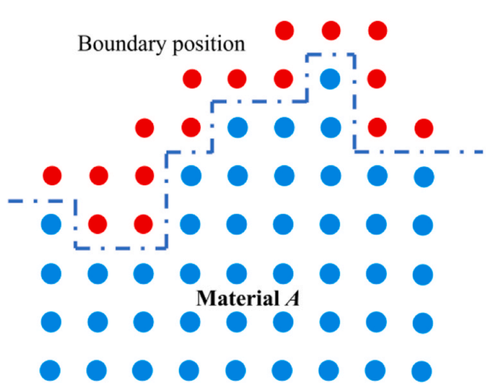

Fig. 33. Complex geometric shapes[148].

4. Extension of peridynamics application

4.1. Analysis of various destructive behavior by peridynamics

(1) Static/quasi-static crack expansion in brittle materials

Unlike classical continuum mechanics, the PD core equation is a spatial integral equation. However, the material points in its discretized system are dynamically changing. This means that PD is not suitable for directly simulating static/quasi-static problems. It should be noted that the brittle materials quasi-static failure process generally includes the crack initiation-expansion process[167]. It often leads to the final overall instability of actual engineering. There are mature methods for dealing with such problems in the numerical methods field. The most common approach is to introduce a dynamic relaxation method with artificial damping. This method can bring the material points into equilibrium under the damping effect. Then, by using the PD advantages in dealing with discontinuities, static/quasi-static problems can be successfully solved.

Huang et al.[57] proposed a local damping introduction idea and a system instability criterion suitable for the PD:

$$\rho\ddot{\mathbf{u}}\left(\mathbf{x}_i,t\right)+C\dot{\mathbf{u}}\left(\mathbf{x}_i,t\right)=\sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\mathbf{f}\left(\mathbf{u}(\mathbf{x}_i,t),\mathbf{u}\left(\mathbf{x}'_j,t\right),\mathbf{x}_i,\mathbf{x}'_j,t\right)v_{c\left(\mathbf{x}'_j\right)}V_{\left(\mathbf{x}'_j\right)}+\mathbf{b}(\mathbf{x}_i,t) \tag{25}$$

where $C\dot{\mathbf{u}}(\mathbf{x}_i,t)$ is the local damping, C is the local damping coefficient, which usually is a positive real constant (kg/m^3s); $\dot{\mathbf{u}}$ is the material point velocity; and the expression $\mathbf{f}\left(\mathbf{u}(\mathbf{x}_i,t),\mathbf{u}\left(\mathbf{x}'_j,t\right),\mathbf{x}_i,\mathbf{x}'_j,t\right)$ is the force density function. Then, by using the external load-graded loading method, the critical loading, crack initiation site, and propagation path were finally accurately predicted under quasi-static conditions. The numerical examples show that the PD with artificial damping can also predict the crack propagation path and fracture mode under quasi-static conditions. Kilic et al.[168] proposed an Adaptive Dynamic Relaxation Method suitable for the PD. By introducing the virtual inertia term and the damping term, the material point velocity and displacement expressions in explicit central difference format were obtained:

$$\mathbf{\Lambda}\ddot{\mathbf{u}}\left(\mathbf{x}_i,t\right)+c_n\mathbf{\Lambda}\dot{\mathbf{u}}\left(\mathbf{x}_i,t\right)=\sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\mathbf{f}\left(\mathbf{u}(\mathbf{x}_i,t),\mathbf{u}\left(\mathbf{x}'_j,t\right),\mathbf{x}_i,\mathbf{x}'_j,t\right)v_{c\left(\mathbf{x}'_j\right)}V_{\left(\mathbf{x}'_j\right)}+\mathbf{b}(\mathbf{x}_i,t) \tag{26}$$

where $\mathbf{\Lambda}$ is the virtual diagonal density matrix, which is determined by Greschgorin's theorem; and c_n is the damping coefficient, which is determined by the Rayleigh quotient. Finally, the numerical example shows that the method has high material damage prediction and numerical convergence. It should be noted that these are the two most representative methods for introducing artificial damping into the PD.

Theoretically, the introduction of local damping into the core equations does not affect the computation of quantitative solutions for static/quasi-static problems, but only accelerates the numerical computation convergence rate.[57] This depends mainly on the selection of the local damping coefficients. In non-linear problems, it is difficult to determine the most effective local damping coefficients (requiring iterative testing), whereas the damping coefficient of the ADRM using quasi-dynamic theory can change adaptively in each cycle, thus avoiding artificial selection.

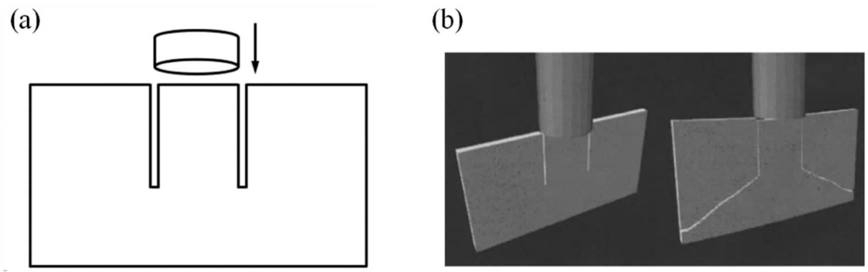

Fig. 34. The Kalthoff-Winkler test: (a) is a two-dimensional schematic diagram; and (b) is the simulation results of Madenci et al.[164].

(2) Dynamic crack extension in brittle materials

Without the introduction of artificial damping for static/quasi-static problems, the material points in the discretized system are dynamic (oscillating near the equilibrium position). This is because the PD core equation is a dynamic form. In the dynamic problem simulation process, artificial damping may lead to non-physical effects. It is generally chosen to ignore the additional damping term in the above content or consider viscous damping. Compared with the static/quasi-static case, the dynamic crack expansion process of brittle materials is shorter and the propagation path is more complex. It is not easy to observe and capture the initiation site and propagation path of dynamic cracks in laboratory tests or field tests. However, PD can reproduce the natural initiation-expansion process of dynamic cracks.

As a classic dynamic fracture test, the Kalthoff-Winkler test has become a classic example of verifying the numerical models' applicability and accuracy in simulating dynamic problems. As shown in Fig. 34, in Dr. Silling's PD theory early stage, Madenci et al. simulated the Kalthoff-Winkler test through PD[164]. This example shows that the failure modes captured by PD are consistent with the laboratory test observation results. Since then, the PD's ability to deal with the dynamic crack initiation-expansion process in various situations is constantly being verified and applied[168].

In theory, the dynamic crack expansion process involves inertial effect and material nonlinearity[169,170]. The dynamic fracture toughness parameters of materials are often related to loading rate, crack expansion speed, and other factors. Considering the dynamic crack expansion inertia effect, scholars have successively carried out the PD model related to deformation rate sensitivity. In 2005, Silling and Askari first proposed the concept of bond-critical elongation considering time variation. Panchadhara et al.[171] proposed a method to estimate the stress intensity factor in PD. By simulating the dynamic crack expansion path process under different loading rates, the correctness and feasibility of the proposed method were finally verified. The numerical examples show that with the increase in loading rate, the crack expansion speed and the number of bifurcations are positively correlated. Zhang et al.[172] simulated the crack expansion process of annular specimens with holes under dynamic loading. The simulation results were compared and the relationship between the hole and the crack expansion path was discussed. The numerical examples show that the BB-PD can simulate the rock-like materials failure process under dynamic loading. Lee et al.[173] proposed a PD model considering different loading rates. By simulating the crack initiation-expansion process of materials prefabricated notched brittle under biaxial tensile conditions, the failure mechanism of brittle materials was analyzed and revealed from multiple perspectives. The numerical examples show that PD can capture the crack failure mode of brittle materials under different loading rates.

Imachi et al.[174] proposed an ordinary SB-PD model based on transition bonds. By establishing a new bond failure criterion, and using damping to test the validity of failure criteria, the crack expansion process in the Kalthoff-Winkler test was reproduced. Zhou et al.[175] proposed a non-ordinary SB-PD model considering the stress failure criterion. By simulating a single crack propagation-bifurcation process in brittle materials under dynamic biaxial conditions, the effects of geometric characteristics and loading conditions were analyzed and summarized. As shown in Fig. 35, the crack growth path of a slab with prefabricated cracks is in good agreement with previous research results. It should be noted that the above two models show that the SB-PD can simulate the brittle materials crack initiation-expansion process under dynamic loading.

(3) The impact fracture and damage behavior in brittle materials

In general, compared with static/quasi-static and dynamic problems, the brittle materials crack expansion process under impact failure is the shortest, the expansion path is the most complex, and the loading capacity loss is large[177–179]. It is difficult to accurately reveal the crack expansion process and failure mechanism of impact fracture. The characteristics that are difficult to observe and capture pose great challenges to the actual engineering safety design. However, due to the PD dynamic form equations, it has been used to study the impact fracture process of many brittle materials. It has shown strong applicability. For example, composite materials[180], layered glass systems[181, 182], and ceramic materials[183].

In Dr. Silling's PD early stage, the applicability of PD to the impact fracture and damage of brittle materials was first studied[184]. Since then, the first PD numerical code EMU has realized the application, promotion, and development of PD theory. It simulated a series of systematic classical examples. For example, bullet penetration, rigid body impact brittle plate, concrete ball impact rigid plate, and other tests[16, 26,58,185]. Demmie and Silling[186,187] simulated the large concrete structure damage process under impact and explosion by EMU and systematically discussed the ultimate loading. Moreover, Demmie and Ostoja Starzewski[188] constructed a PD model suitable for studying the geological materials' fracture and damage process under impact loading. This means that PD has the potential to become a new numerical simulation method for large engineering structures' dynamic catastrophic failure analysis.

The PD is making progress in the study of brittle materials impact

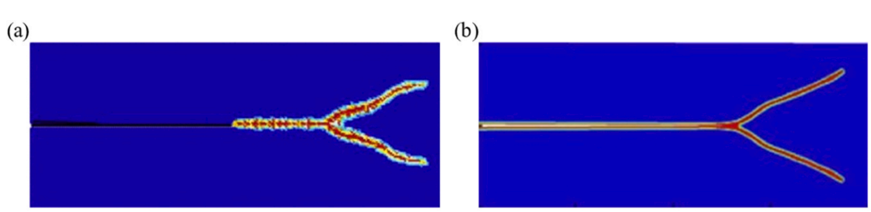

Fig. 35. Comparison of crack branching[175]: (a) non-ordinary SB-PD model[175]; and (b) BB-PD model by Ha and Bobaru[176].

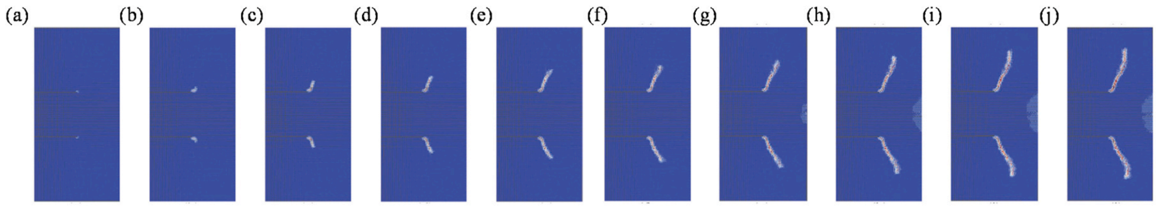

Fig. 36. Crack initiation, growth, and propagation in the simulation of the Kalthoff-Winkler test[189]: (a) 25 ms; (b) 30 ms; (c) 35 ms; (d) 40 ms; (e) 45 ms; (f) 50 ms; (g) 55 ms; (h) 60 ms; (i) 65 ms; (j) 70 ms.

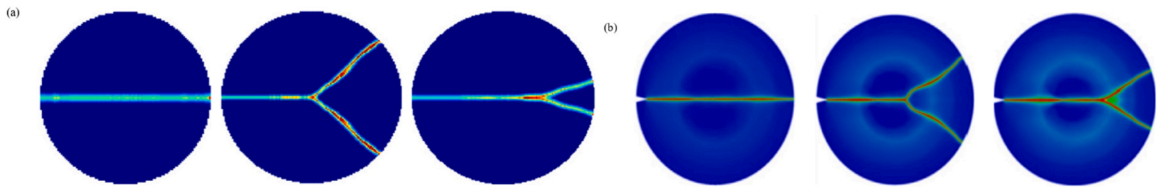

Fig. 37. Comparisons of ultimate thermal shock crack patterns[194]: (a) the PD model[194]; and (b) the phase-field model.

damage process. Guo et al.[189] proposed an ordinary SB-PD model considering the linear constitutive model. By simulating the Kalthoff-Winkler test under different low-impact rates, the correctness of this method was finally verified, as shown in Fig. 36. Wang et al.[190] proposed a non-ordinary SB-PD thermo-visco-plastic model. By simulating the Kalthoff-Winkler test under different impact velocities, the variation of crack expansion angle and speed under different impact velocities was finally obtained. Wu et al.[191] proposed a PD model considering bond dynamic fracture. By establishing the relationship between the bond dynamic tensile/compressive damage and the bond elongation, the crack expansion path and speed were finally accurately predicted.

In addition, the continuous development of the PD thermo-mechanics coupling model also shows strong applicability in the study of brittle materials' thermal shock failure process. Meanwhile, PD can reveal the mechanism of the crack initiation-expansion process under thermal shock to a certain extent. This is usually a disadvantage that traditional numerical methods cannot overcome[192,193]. Wang et al.[194] simulated the brittle materials' crack expansion process under thermal shock and infrared radiation heating. The influence of crack tip strain energy, bifurcation angle, and expansion speed can be obtained. As shown in Fig. 37, the comparison between the PD results and the phase-field results under three different thermal shocks shows a good agreement. Wang et al.[195] proposed an improved BB-PD thermo-mechanics coupling model. By considering the relationship between the material points' relative displacement and the thermal elastic stiffness, the ceramic materials crack expansion test under quenching conditions was simulated. Finally, the brittle materials' damage mechanism under thermal shock was analyzed. D′antuono et al.[196] proposed an ordinary SB-PD thermo-mechanics coupling model. By simulating the fracture of ceramic thin plates and thick plates under thermal shock, the fracture morphology of two-dimensional parallel cracks and three-dimensional columnar honeycomb cracks was finally found.

4.2. Various peridynamics coupling methods

As mentioned above, the continuity assumption of traditional classical continuum mechanics makes a material point only interact and exchange information with its adjacent points. When dealing with the crack initiation-expansion process, this continuous displacement field and stress field will produce singularity. As the most representative numerical method based on the continuity hypothesis, FEM has achieved great success in structural deformation analysis[197–199]. However, it is difficult to solve the materials' fracture and damage. FEM solves the singularity problem of displacement field and stress field by introducing additional fracture criteria. However, it will face the mesh dependence problem. The Extended Finite Element Method (XFEM) solves the displacement discontinuity problem by introducing step-shape functions with discontinuous properties. However, it will face the problem that multi-crack expansion requires more criteria. Meanwhile, PD has obvious advantages in dealing with discontinuous problems. It can simulate the crack natural initiation-expansion process and reflect the multi-crack interaction process. It should be noted that the above PD advantages do not mean that it is completely superior to

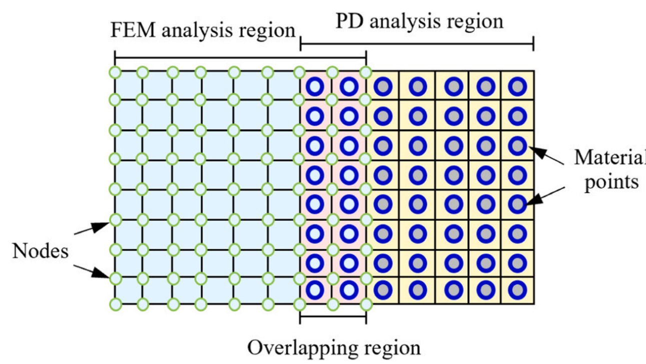

Fig. 38. PD-FEM analysis region division[53].

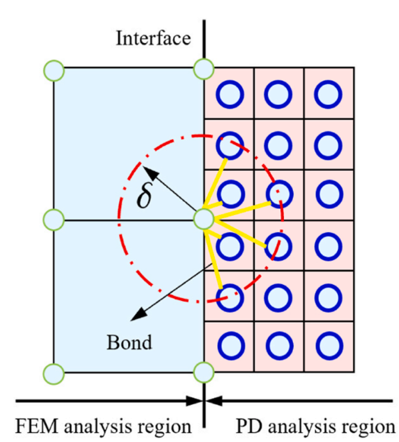

Fig. 39. PD-FEM interface element[200].

traditional numerical methods, such as FEM and XFEM. The non-local boundary conditions and PD surface effects are caused by its non-local theory. Meanwhile, the PD computational efficiency is not high, which is especially reflected in the SB-PD. This makes various PD coupling methods with their advantages have been proposed.

In recent years, because of the high efficiency of FEM in dealing with continuous problems and the advantages of PD in dealing with discontinuous problems, the PD-FEM method has gradually attracted attention [53]. In general, this kind of coupling method mainly divides the structure into PD analysis region, FEM region, and overlapping region, as shown in Fig. 38. Then, the PD discrete system is used to deal with the crack initiation-expansion region, and the FEM is used to deal with other continuous regions that do not require mang calculation. Not only the PD surface effect can be alleviated, but also the computational efficiency can be greatly improved, and the discontinuous problem can be better handled.

The key problem of this method is how to solve the interface data transmission problem. For example, Kilic et al.[163] used the interpolation function to realize the interface data transmission by setting the overlapping area of PD and FEM analysis regions. Liu et al.[200] introduced the interface element into PD-FEM, and finally used the material point force density to calculate the interface element node force, as shown in Fig. 39. The breakthrough of the key problem has promoted the development and application of this method. Zhou et al. [201] proposed a fast way of solving static problems based on PD-FEM. The discrete PD equation of motion is rewritten as

$$\begin{Bmatrix} \underline{\ddot{U}}^{P}_{n} \\ \underline{\underline{\ddot{U}}}^{P}_{n} \end{Bmatrix} + c_n \begin{Bmatrix} \underline{\dot{U}}^{P}_{n} \\ \underline{\underline{\dot{U}}}^{P}_{n} \end{Bmatrix} = \begin{bmatrix} \underline{D}^{-1} & 0 \\ 0 & \underline{\underline{D}}^{-1} \end{bmatrix} \begin{Bmatrix} \underline{F}^{P}_{n} \\ \underline{\underline{F}}^{P}_{n} \end{Bmatrix} \tag{27}$$

where U^p is the displacement vector; F^p is the unbalanced force vector of the PD sub-region, and the superscript indicates variables related to the PD sub-region; c_n is the damping coefficient at the n^{th} time increment; and D is a virtual diagonal density matrix. Similarly, the FEM equation can be expressed as

$$\begin{Bmatrix} \underline{\ddot{U}}^{f}_{n} \\ \underline{\underline{\ddot{U}}}^{f}_{n} \end{Bmatrix} + c_n \begin{Bmatrix} \underline{\dot{U}}^{f}_{n} \\ \underline{\underline{\dot{U}}}^{f}_{n} \end{Bmatrix} = \begin{bmatrix} \underline{M}^{-1} & 0 \\ 0 & \underline{\underline{M}}^{-1} \end{bmatrix} \begin{Bmatrix} \underline{F}^{f}_{n} \\ \underline{\underline{F}}^{f}_{n} \end{Bmatrix} \tag{28}$$

where the superscript indicates variables related to the FEM sub-region, and M is the diagonal mass matrix. Then, by coupling the interface element sub-region, using the coupling model global matrix and ADRM, the elastic stage static solution and the crack initiation-expansion process can be obtained quickly. The numerical example shows that this result of the coupling method is very agreement compared with the single PD, as shown in Fig. 40.

Yang et al.[202] proposed an ordinary SB-PD and FEM coupling method. By considering the balance of forces between different meshes, the complexity of the coupling model transition region is avoided. Meanwhile, the brittle materials crack initiation-expansion process under dynamic loading was simulated. It not only effectively alleviates the PD surface effect, but also verifies the correctness of the proposed coupling model. It should be noted that the PD-FEM is not suitable for simulating the temperature field discontinuous variation during crack expansion under thermal shock. Zaccariotto et al.[203] proposed an effective method to couple FEM meshes to PD meshes, i.e., an adaptive algorithm that allows transforming FEM nodes into PD points, and used it in static and dynamic conditions. The new method solved the problem of crack propagation with crack branching. Ni et al.[204] proposed two implicit static solution procedures to study crack propagation by BB-PD. The discretized structures in space by a coupled PD-FEM approach (exploiting the flexibility of FEM) can reduce the computational cost of simulation. Then, Ni et al.[205] proposed a method to couple FEM nodes with SB-PD points, which has the efficiency of FEM and the flexibility of PD in dealing with crack propagation. Meanwhile, it can remove the PD surface effects of OSB-PD.

It should be noted that many methods can be coupled with PD other than FEM. For example, MD[206], phase field theory[207,208], SPH [209], and the Reproducing Kernel Particle Method[210]. Corresponding developments have been made in various coupling methods. For example, Ganzenmüller et al.[211] carried out a similarity study between PD and SPH; Ren et al.[212] carried out a hybrid modeling study of SB-PD and SPH; Tong et al.[213] carried out a multi-scale coupling analysis study of SB-PD and MD.

In summary, the ability of PD to solve discontinuous problems in various situations is constantly improving and breaking through. Firstly, by introducing artificial damping, the material points in the discretized system reach equilibrium under damping. This makes the PD show strong applicability in dealing with static/quasi-static problems. Then, the PD integral core equation is a dynamic form. This makes PD show strong universality in dealing with dynamic problems and impact problems. It can even make up for the shortcomings that are difficult to observe or capture in laboratory tests or in-situ tests. Finally, the coupling methods, such as PD-FEM, are showing strong development

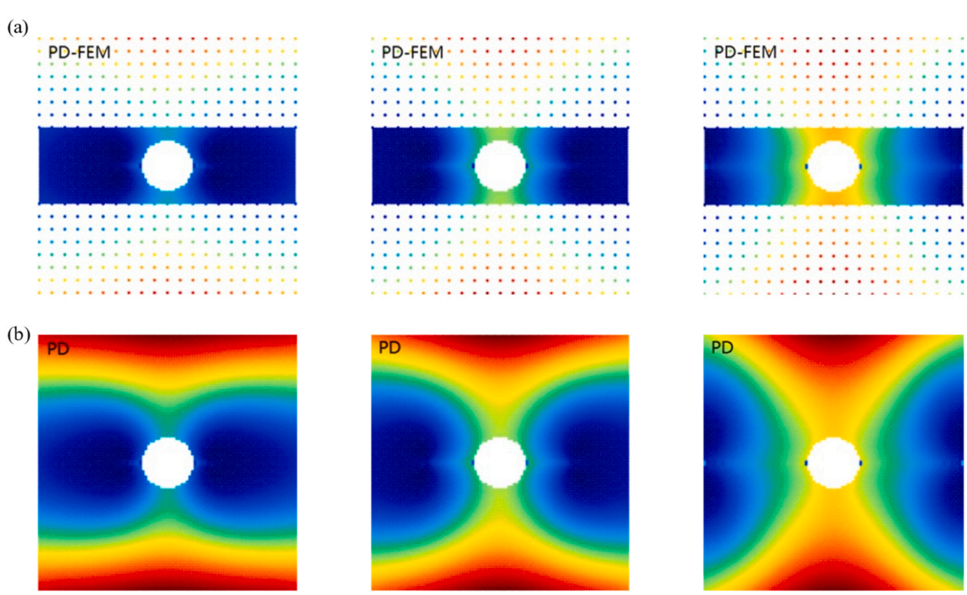

Fig. 40. Initiation, evolution, and fracture process of a plate with a circular hole under tensile action[201]: (a) PD-FEM[201]; and (b) single PD.

potential and application prospects. It can effectively alleviate the PD surface effect, and reduce the computational cost. It can also solve the drawbacks of classical numerical simulation methods when dealing with discontinuous problems.

5. Conclusion and Prospect

The PD theoretical framework, as non-local method, has been greatly improved in more than 20 years of development. It has been successfully applied to analyze the deformation, damage, and fracture processes of various solid materials or structures at the macro/micro-scale. It has been also successfully applied in many other research fields. The PD can show the advantages different from the other numerical methods in dealing with discontinuous problems since it is based on non-local theory and spatial integral equation. It is showing great potential as a research method for evaluating the complicated mechanical behaviors of solids although it is not completely superior to the existing numerical methods.

The PD is still in the preliminary stage, and its potential has not been fully developed. The continuous development of PD is almost aimed at expanding its scope of application. In theory, most PD theoretical studies are devoted to enriching its simulation capability. The PD has been gradually developed to expand its capability from the analysis of linear elastic isotropic materials and type-I cracks to the analysis of elastic-plastic anisotropic materials, type-II cracks, and type-III cracks.

In the damage and fracture mechanics field, quantitative analysis of type-II and type-III cracks, the elastic-plastic constitutive model considering coupling factors, and the multi-scale characterization of classical physical quantities at the crack tip are to be studied by the PD methods. Meanwhile, attention should be paid to improve the computing accuracy of PD at the methodological level to overcome the simulation problems, such as the numerical approximation problems caused by spatial discretization, the lack of more effective PD surface effect correction algorithms, and so on.

In application, most PD-based research is devoted to the diversification of fracture forms and materials. The current research mostly focuses in the study of mechanical or physical properties of solids in the laboratory scale. There are very rare research investigating the actual engineering scale due to computing costs. The refined numerical model needs to consider computational efficiency since the non-local theoretical characteristics of PD will inevitably lead to high computational costs. How to improve the computational performance is a big challenge for PD to be an excellent numerical method.

Moreover, when facing actual engineering structures or materials in complicated environmental conditions, accurate modeling and multi-disciplinary coupling methods of PD are still in its primary stage. The development of multi-field coupling models will greatly expand the application of PD in multi-medium problems of different disciplines. With the deepening of research, the PD method will attract new developments in different fields to become ultimately a reliable and efficient numerical method.

CRediT authorship contribution statement

Feng Tian: Investigation, Methodology, Writing – original draft, Writing – review & editing. **Jinxin Zhou:** Methodology, Writing – original draft. **Jian-fu Shao:** Writing – review & editing. **Zaobao Liu:** Funding acquisition, Investigation, Methodology, Supervision, Writing – review & editing. **Hongbo Li:** Methodology, Writing – original draft. **Enda Zhang:** Methodology, Writing – original draft.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

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